Loewner's Theory: a parametric method to tackle problems in Complex Analysis

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Schedule of the lectures.

- The Bieberbach Conjecture
  1. The area theorem and the case $n = 2$ in Bieberbach Conjecture
  2. Some applications: distortion theorems
- Radial Loewner Theory
  1. First steps in Loewner theory
  2. From Loewner chains to Loewner PDE and the case $n = 3$ in Bieberbach conjecture
  3. From Loewner PDE to Loewner chains: evolution families
  4. Some applications: Univalence criteria
- Chordal Loewner Theory
- Semigroups of analytic functions
- General Loewner Theory
- Some open problems

First steps in Loewner theory

Definition

A family of holomorphic functions in $\mathbb{D}$, $(f_t)_{t\geq 0}$, is said to be a *(radial) Loewner chain*, if

LC1 all f_t's are univalent in $\mathbb{D}$;

LC2 $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leq s \leq t$;

LC3 for any $t \geq 0$, $f_t(z) = e^t z + a_2(t)z^2 + \ldots$, i.e., $e^{-t} f_t \in \mathcal{S}$.

Theorem

For any $f \in \mathcal{S}$, there exists a Loewner chain (f_t) such that $f = f_0$.

Theorem

Let (f_t) be a Loewner chain. Then there exists a Herglotz function p such that $\frac{\partial f_t(z)}{\partial t} = z f_t'(z) p(z, t)$.

First steps in Loewner theory

Definition

We say that $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is a Herglotz function if

1. $z \mapsto p(z, t)$ is analytic for all t;
2. $t \mapsto p(z, t)$ is measurable for all z;
3. $p(0, t) = 1$ for all t;
4. $\operatorname{Re} p(z, t) > 0$ for all z and all t.

From Loewner PDE to Loewner chains

In the section we prove that any Loewner PDE has a solution which is a Loewner chain.

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We apply this result to get univalence criteria for analytic maps.

From Loewner PDE to Loewner chains

Theorem

Let $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ be a Herglotz function. Then, for any $z \in \mathbb{D}$ and any $s \geq 0$ the initial value problem

$$\frac{dw}{dt} = -wp(w,t) \quad \text{for a.e. } t \in [s, +\infty), \qquad w(s) = z \qquad \text{(Loewner ODE)}$$

has a unique solution $w(t)$ in $t \in [s, +\infty)$.

From Loewner PDE to Loewner chains

Theorem

Let $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ *be a Herglotz function. Then, for any* $z \in \mathbb{D}$ *and any* $s \geq 0$ *the initial value problem*

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(Loewner ODE)

has a unique solution $w(t)$ *in* $t \in [s, +\infty)$*. Write* $\varphi_{s,t}(z) := w(t)$*. Then* $\varphi_{s,t}$ *is a univalent for all* $0 \leq s \leq t < +\infty$ *and*

EF1 $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$,

EF2 $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ *whenever* $0 \leq s \leq u \leq t < +\infty$,

EF3 $\varphi_{s,t}(0) = 0$ *and* $\varphi'_{s,t}(0) = e^{s-t}$.

Definition

A family of analytic self-maps of the unit disk $(\varphi_{s,t})_{0 \leq s \leq t < +\infty}$ satisfying EF1, EF2, and EF3 is called an evolution family.

From Loewner PDE to Loewner chains

Proof.

We define $w_0(z,t) \equiv 0$ and, recursively,

$$w_{n+1}(z,t) = z \exp\left[-\int_s^t p(w_n(z,\tau),\tau)d\tau\right] \quad n = 0,1,2,3...$$

for $t \in [s,+\infty)$ and for all $z \in \mathbb{D}$. Notice that $|w_n(z,t)| < 1$ for all n and the function $z \mapsto w_n(z,t)$ is analytic for all t.

From Loewner PDE to Loewner chains

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for $t \in [s,+\infty)$ and for all $z \in \mathbb{D}$. Notice that $|w_n(z,t)| < 1$ for all n and the function $z \mapsto w_n(z,t)$ is analytic for all t.

Schwarz's lemma implies that $|w_n(z,t)| < |z|$.

From Loewner PDE to Loewner chains

Proof. Using again the representation formula $p(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)$ we deduce that

$$|p'(z,\tau)| \leq 2(1-|z|)^{-2}.$$

From Loewner PDE to Loewner chains

Proof. Using again the representation formula $p(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)$ we deduce that

$$|p'(z,\tau)| \leq 2(1-|z|)^{-2}.$$

Moreover $|e^{-a} - e^{-b}| \leq |a-b|$ whenever $\operatorname{Re} a, \operatorname{Re} b \geq 0$. Thus

$$|w_{n+1}(z,t) - w_n(z,t)| =$$

$$= |z| \left| \exp\left[-\int_s^t p(w_n(z,\tau),\tau)d\tau\right] - \exp\left[-\int_s^t p(w_{n-1}(z,\tau),\tau)d\tau\right]\right|$$

$$\leq \int_s^t |p(w_n(z,\tau),\tau) - p(w_{n-1}(z,\tau),\tau)|\, d\tau$$

$$= \int_s^t \left| \int_{w_{n-1}(z,\tau)}^{w_n(z,\tau)} p'(\xi,\tau)d\xi \right| d\tau \leq \int_s^t \int_{w_{n-1}(z,\tau)}^{w_n(z,\tau)} \left| \frac{2}{(1-|\xi|)^2} \right| d\xi d\tau$$

$$\leq \int_s^t \int_{w_{n-1}(z,\tau)}^{w_n(z,\tau)} \left| \frac{2}{(1-|z|)^2} \right| d\xi d\tau \leq \frac{2}{(1-|z|)^2} \int_s^t |w_n(z,\tau) - w_{n-1}(z,\tau)|\, d\tau.$$

From Loewner PDE to Loewner chains

Proof.

$$|w_{n+1}(z,t) - w_n(z,t)| \leq \frac{2}{(1-|z|)^2} \int_s^t |w_n(z,\tau) - w_{n-1}(z,\tau)|\, d\tau.$$

By induction we have that

$$|w_{n+1}(z,t) - w_n(z,t)| \leq \frac{2^n (t-s)^n}{(1-|z|)^{2n} n!} \quad (z \in \mathbb{D},\ n = 0, 1, 2, 3, \ldots).$$

We conclude that $\lim_n w_n(z,t)$ exists uniformly in $|z| \leq r$, $s \leq t \leq T$ for every $r < 1$ and T.

From Loewner PDE to Loewner chains

Proof.

$$|w_{n+1}(z,t) - w_n(z,t)| \leq \frac{2}{(1-|z|)^2}\int_s^t |w_n(z,\tau) - w_{n-1}(z,\tau)|\,d\tau.$$

By induction we have that

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We conclude that $\lim_n w_n(z,t)$ exists uniformly in $|z| \leq r$, $s \leq t \leq T$ for every $r < 1$ and T.
Denoting the limit by $\varphi_{s,t}(z)$, we obtain an analytic function in z and by the Lebesgue's bounded convergence theorem we obtain that

$$\varphi_{s,t}(z) = z\exp\left[-\int_s^t p(\varphi_{s,\tau}(z),\tau)d\tau\right].$$

From this equation, it is easy to deduce the properties of $\varphi_{s,t}$.

From Loewner PDE to Loewner chains

Corollary

Take p and $(\varphi_{s,t})$ as in the above theorem. Then

$$f_s(z) = \lim_{t\to+\infty} e^t \varphi_{s,t}(z)$$

exists uniformly on compacta of $\mathbb{D}$, (f_t) is a Loewner chain satisfying $f_s = f_t \circ \varphi_{s,t}$ and

$$\frac{\partial f_t(z)}{\partial t} = z f_t'(z) p(z,t).$$

From Loewner PDE to Loewner chains

Proof.

We just obtain the existence of the limit $f_s(z) = \lim_{t\to+\infty} e^t \varphi_{s,t}(z)$ and $f_s = f_t \circ \varphi_{s,t}$. The rest of the proof is easy.

By the proof of the above theorem, we have that

$$\varphi_{s,t}(z) = z \exp\left[-\int_s^t p(\varphi_{s,\tau}(z),\tau)d\tau\right].$$

Therefore

$$e^{t-s}\varphi_{s,t}(z) = z \exp\left[\int_s^t (1 - p(\varphi_{s,\tau}(z),\tau))d\tau\right]$$

and this function belongs to $\mathcal{S}$. By Koebe distortion theorem, we have

$$|\varphi_{s,t}(z)| \le \frac{|z|}{(1-|z|)^2} e^{s-t}.$$

From Loewner PDE to Loewner chains

Proof.

Fix $r<1$. If $|z|<r$ then $|\varphi_{s,t}(z)|\leq r$ and

$$|1-p(\varphi_{s,\tau}(z),\tau)|=\left|\int_0^{\varphi_{s,\tau}(z)}p'(\xi,\tau)d\xi\right|\leq\frac{2}{(1-r)^2}|\varphi_{s,\tau}(z)|\leq\frac{2}{(1-r)^4}e^{s-\tau}.$$

Thus, if $|z|\leq r$ and $t,t'\geq s$, after some computations, we have

$$|e^{t-s}\varphi_{s,t}(z)-e^{t'-s}\varphi_{s,t'}(z)|=\ldots=$$

$$=|e^{t-s}\varphi_{s,t}(z)|\left|1-\exp\left[\int_t^{t'}(1-p(\varphi_{s,\tau}(z),\tau))d\tau\right]\right|$$

$$\leq\frac{|z|}{(1-|z|)^2}\left|1-\exp\left[\int_t^{t'}(1-p(\varphi_{s,\tau}(z),\tau))d\tau\right]\right|.$$

Thus the family $(e^{t-s}\varphi_{s,t})_{t\geq s}$ is Cauchy in the compact set $r\overline{\mathbb{D}}$.

From Loewner PDE to Loewner chains

Proof.

Thus the limit

$$f_s(z) = \lim_{t \to +\infty} e^t \varphi_{s,t}(z)$$

is well defined. Moreover, if $s < t$,

$$f_s(z) = \lim_{\tau \to +\infty} e^\tau \varphi_{s,\tau}(z) = \lim_{\tau \to +\infty} e^\tau \varphi_{t,\tau} \circ \varphi_{s,t}(z) = f_t(\varphi_{s,t}(z)).$$

Univalence

So far, we have applied Loewner theory just in one direction: if (f_t) is a Loewner chain then it is the solution of a certain PDE

$$\frac{\partial f_t(z)}{\partial t} = z f_t'(z) p(z,t).$$

Moreover, we have learned that such PDE has a solution which is a Loewner chain.

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But it can have other solutions: take h an entire function and write $g_t := h \circ f_t$. Then

$$\frac{\partial g_t(z)}{\partial t} = h'(f_t(z))\frac{\partial f_t(z)}{\partial t} = h'(f_t(z)) z f_t'(z) p(z,t) = z g_t'(z) p(z,t).$$

The functions g_t are not univalent unless $h(z) = az + b$.

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The goal of this section is to characterize Loewner chains between all the solution of the above PDE and, as a by product of such result, we plan to get univalence criteria.

Univalence

Theorem (Univalence Theorem)

Let $0 < r < 1$. Let (f_t) be a family of analytic maps in $r\mathbb{D}$ s.t.

1. $f_t(0) = 0$ *and* $f_t'(0) = e^t$;
2. *For each* $z \in r\mathbb{D}$, *the map* $t \mapsto f_t(z)$ *is abs. continuous;*
3. *There is* K *such that*

$$|f_t(z)| \leq Ke^t \quad \text{for all } |z| < r, \text{ and } t \geq 0;$$

4. *There exists a Herglotz function* $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ *s.t.*

$$\frac{\partial f_t(z)}{\partial t} = zf_t'(z)p(z, t) \quad \text{for all } |z| < r, \text{ and } t \geq 0.$$

Then the function f_t *has an analytic continuation to* $\mathbb{D}$ *and* (f_t) *is a Loewner chain.*

Univalence

Proof. Take $(\varphi_{s,t})$ the evolution family associated with p. Fix $z \in r\mathbb{D}$.
Using again Vitali's theorem, $t \mapsto k(t) := f_t(\varphi_{s,t}(z))$ is absolutely continuous in every interval of the form $[s, T]$. Then

$$\begin{aligned} k'(t) &= f_t'(\varphi_{s,t}(z))\frac{\partial}{\partial t}(\varphi_{s,t}(z)) + \frac{\partial f_t}{\partial t}(\varphi_{s,t}(z)) \\ &= -f_t'(\varphi_{s,t}(z))\varphi_{s,t}(z)p(\varphi_{s,t}(z), t) \\ &\qquad +\varphi_{s,t}(z)f_t'(\varphi_{s,t}(z))p(\varphi_{s,t}(z), t) = 0. \end{aligned}$$

But $k(s) = f_s(z)$. Then $f_s(z) = k(s) = k(t) = f_t \circ \varphi_{s,t}(z)$ whenever $|z| < r$.

...

Univalence

Proof. ... By Schwarz's lemma (applied twice), if $h : \mathbb{D} \to \mathbb{D}$ is analytic and $h(0) = h'(0) = 0$, then $|h(w)| \leq |w|^2$ for all $w \in \mathbb{D}$.

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Write $h(w) = \frac{e^{-t} f_t(rw) - rw}{K+1}$. By hypothesis, we have h is an analytic self-map of $\mathbb{D}$ such that $h(0) = h'(0) = 0$. Thus, taking $z = rw$ we have

$$|e^{-t} f_t(z) - z| \leq (K+1) r^{-2} |z|^2.$$

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Hence, using this inequality and applying Koebe distortion theorem to the function $e^{s-t}\varphi_{s,t}$, we have

$$\begin{aligned} |f_s(z) - e^t \varphi_{s,t}(z)| &= e^t |e^{-t} f_t(\varphi_{s,t}(z)) - \varphi_{s,t}(z)| \\ &\leq e^t (K+1) r^{-2} |\varphi_{s,t}(z)|^2 \\ &\leq e^t (K+1) r^{-2} |z|^2 (1 - |z|)^{-4} e^{2s-2t} \\ &\leq (K+1)(1-r)^{-4} e^{2s-t}. \end{aligned}$$

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$$\begin{aligned}
|f_s(z)-e^t\varphi_{s,t}(z)| &= e^t|e^{-t}f_t(\varphi_{s,t}(z))-\varphi_{s,t}(z)| \\
&\leq e^t(K+1)r^{-2}|\varphi_{s,t}(z)|^2 \\
&\leq e^t(K+1)r^{-2}|z|^2(1-|z|)^{-4}e^{2s-2t} \\
&\leq (K+1)(1-r)^{-4}e^{2s-t}.
\end{aligned}$$

Therefore, $e^t\varphi_{s,t}(z)\to f_s(z)$ as $t\to\infty$ for $|z|<r$.

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Write $h(w)=\frac{e^{-t}f_t(rw)-rw}{K+1}$. By hypothesis, we have h is an analytic self-map of $\mathbb{D}$ such that $h(0)=h'(0)=0$. Thus, taking $z=rw$ we have

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Therefore, $e^t\varphi_{s,t}(z)\to f_s(z)$ as $t\to\infty$ for $|z|<r$.

This implies that f_s is nothing but the restriction to $r\mathbb{D}$ of the function built in the last corollary.

Starlike functions

A function f is called spirallike of type $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ if it is univalent in $\mathbb{D}$ and $\exp(-e^{i\alpha}t)f(\mathbb{D}) \subseteq f(\mathbb{D})$ for all $t \geq 0$. Starlike functions are the special case $\alpha = 0$.

Corollary (Spacek, 1933, Robertson, 1961)

The function $f(z) = z + \ldots$ analytic in $\mathbb{D}$ is spirallike of type α if and only if

$$\mathrm{Re}\left[e^{i\alpha} z \frac{f'(z)}{f(z)}\right] > 0 \quad \textit{for all } z \in \mathbb{D}.$$

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Proof. Define $f_t(z) = e^t f(z) = e^t z + \ldots$ for all $z \in \mathbb{D}$ and $t \geq 0$.
Since

$$\frac{\partial f_t(z)/\partial t}{z f_t'(z)} = \frac{e^t f(z)}{z e^t f'(z)} = \frac{f(z)}{z f'(z)},$$

defining $p(z,t) = \frac{f(z)}{zf'(z)}$ it is a non-negative function for which

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Clearly (f_t) satisfies the hypothesis of Univalence Theorem and f_t is a Loewner chain. This implies that $f = f_0$ is univalent and $f(\mathbb{D}) = f_0(\mathbb{D}) \subseteq f_t(\mathbb{D}) = e^t f(\mathbb{D})$, that is, $e^{-t} f(\mathbb{D}) \subseteq f(\mathbb{D})$ for all t.

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Proof. Define $f_t(z) = e^t f(z) = e^t z + \dots$ for all $z \in \mathbb{D}$ and $t \geq 0$. Since

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Univalence criteria

From distortion theorem, one can prove that any $f \in \mathcal{S}$ satisfies

$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| < 6 \qquad (z \in \mathbb{D}).$$

The converse is not true.
Take $f(z) = \exp(\lambda z)$, with $\lambda > \pi$. f is not univalent in $\mathbb{D}$ and

$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| \leq \frac{2\sqrt{3}\lambda}{9}, \qquad \frac{2\sqrt{3}\pi}{9} \approx 1.209...$$

But

Corollary (Becker, 1972)

Let f be analytic in $\mathbb{D}$ with $f'(0) \neq 0$. If

$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| < 1 \qquad (z \in \mathbb{D})$$

then f is univalent in $\mathbb{D}$.

Univalence criteria

Proof. We may assume that $f'(0) = 1$. Let

$$f_t(z) := f(e^{-t}z) + (e^t - e^{-t})zf'(e^{-t}z) \quad \text{for all } z \in \mathbb{D} \text{ and } t \geq 0.$$

Then

$$\frac{\partial f_t(z)}{\partial t} = e^t zf'(e^{-t}z) - (1 - e^{-2t})z^2 f''(e^{-t}z),$$
$$zf_t'(z) = e^t zf'(e^{-t}z) + (1 - e^{-2t})z^2 f''(e^{-t}z).$$

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$$zf_t'(z) = e^t zf'(e^{-t}z) + (1 - e^{-2t})z^2 f''(e^{-t}z).$$

Write $p(z,t) := \frac{\partial f_t(z)}{\partial t}/(zf_t'(z))$. Then

$$\left|\frac{p(z,t)-1}{p(z,t)+1}\right| = \left|\frac{\frac{\partial f_t(z)}{\partial t} - zf_t'(z)}{\frac{\partial f_t(z)}{\partial t} + zf_t'(z)}\right| =$$

$$= (1 - e^{-2t})\left|e^{-t}z\frac{f''(e^{-t}z)}{f'(e^{-t}z)}\right| \leq \frac{1 - e^{-2t}}{1 - e^{-2t}|z|^2} < 1.$$

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$$f_t(z) := f(e^{-t}z) + (e^t - e^{-t})zf'(e^{-t}z) \quad \text{for all } z \in \mathbb{D} \text{ and } t \geq 0.$$

Then

$$\frac{\partial f_t(z)}{\partial t} = e^t zf'(e^{-t}z) - (1 - e^{-2t})z^2 f''(e^{-t}z),$$
$$zf_t'(z) = e^t zf'(e^{-t}z) + (1 - e^{-2t})z^2 f''(e^{-t}z).$$

Write $p(z,t) := \frac{\partial f_t(z)}{\partial t} / (zf_t'(z))$. Then

$$\left|\frac{p(z,t)-1}{p(z,t)+1}\right| = \left|\frac{\frac{\partial f_t(z)}{\partial t} - zf_t'(z)}{\frac{\partial f_t(z)}{\partial t} + zf_t'(z)}\right| =$$

$$= (1 - e^{-2t})\left|e^{-t}z\frac{f''(e^{-t}z)}{f'(e^{-t}z)}\right| \leq \frac{1 - e^{-2t}}{1 - e^{-2t}|z|^2} < 1.$$

The map $T(z) = \frac{z+1}{1-z}$ sends the unit disk into the right half-plane. But $p(z,t) = T\left(\frac{p(z,t)-1}{p(z,t)+1}\right)$.

Univalence criteria

Proof. We may assume that $f'(0) = 1$. Let

$$f_t(z) := f(e^{-t}z) + (e^t - e^{-t})zf'(e^{-t}z) \quad \text{for all } z \in \mathbb{D} \text{ and } t \geq 0.$$

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$$\frac{\partial f_t(z)}{\partial t} = e^t zf'(e^{-t}z) - (1 - e^{-2t})z^2 f''(e^{-t}z),$$

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The map $T(z) = \frac{z+1}{1-z}$ sends the unit disk into the right half-plane. But $p(z,t) = T\left(\frac{p(z,t)-1}{p(z,t)+1}\right)$. Therefore $\operatorname{Re} p(z,t) > 0$. ...

Univalence criteria

Proof. ...
Since $\frac{\partial f_t(z)}{\partial t} = zf_t'(z)p(z,t)$, the family (f_t) satisfies the hypothesis of Univalence Theorem.

Thus (f_t) is a Loewner chain and, in particular, $f = f_0$ is univalent.

Radial Loewner theory

Summing up, we have introduced three concepts: evolutions families, Herglotz functions and Loewner chains.
There is a one to one correspondence between these notions:

Loewner chains (f_t) $\overset{\varphi_{s,t}=f_t^{-1}\circ f_s}{\longleftrightarrow}$ Evolution families $(\varphi_{s,t})$

Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = zp(z,t)f_t'(z)$$

Loewner ODE

$$\frac{dw}{dt} = -zp(w,t), \; w(s) = z$$

$$\varphi_{s,t}(z) := w(t)$$

Herglotz function $p(w,t)$

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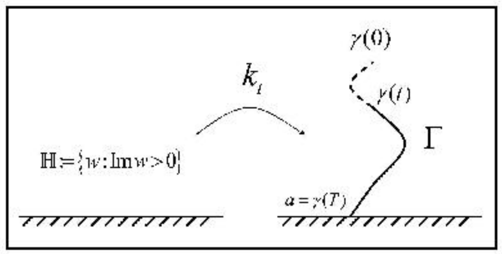
Loewner's theory has been used to prove several deep results in various branches of mathematics, even apparently unrelated to complex analysis. So far, we have presented two of them but there are many other. See Duren and Pommerenke.

Chordal Loewner Theory.

Consider a Jordan arc Γ contained in $\mathbb{H} = \{w : \operatorname{Im} w > 0\}$ except for its endpoint $a \in \mathbb{R}$ parameterized by $\gamma : [0, T] \to \mathbb{H} \cup \{a\}$.
Write $\Omega_t := \mathbb{H} \setminus \gamma([t, T])$, $t \geq 0$.



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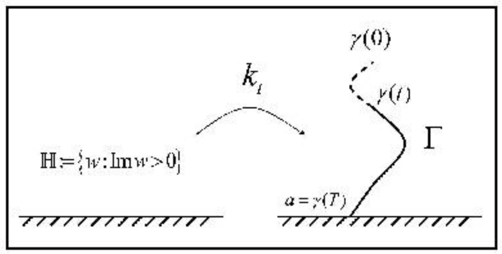
Write $\Omega_t := \mathbb{H} \setminus \gamma([t, T])$, $t \geq 0$.
Then there is a unique univalent and onto $k_t : \mathbb{H} \to \Omega_t$ such that

$$k_t(z) = z - \frac{c(t)}{z} + o(1/z^2), \quad c(t) \geq 0, z \to \infty.$$

Up to a reparametrization we may assume that $c(t) = T - t$.



Chordal Loewner Theory.

Under above normalization, Kufarev, Sobolev and Sporysheva (1968) showed that k_t satisfies the following partial differential equation:

$$\frac{\partial k_t(w)}{\partial t} = -k_t'(w)\frac{2}{w - h(t)}, \qquad (w \in \mathbb{H}),$$

where h is a continuous real-valued function called "driving term".

Chordal Loewner Theory.

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Take $T : \mathbb{D} \to \mathbb{H}$ the biholomorphism $T(z) = i\frac{z+1}{1-z}$.
Its inverse is $T^{-1}(w) = \frac{w-i}{w+i}$. Then $f_t := T^{-1} \circ k_t \circ T$ is analytic in the unit disk. By the chain rule,

$$\frac{\partial f_t(z)}{\partial t} = (T^{-1})'(k_t(z))\frac{\partial k_t(T(z))}{\partial t};$$
$$f_t'(z) = (T^{-1})'(k_t(z))k_t'(T(z))T'(z).$$

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To compare this PDE with the Loewner PDE, we should rewrite them in the same setting, for example the unit disk.
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$$\begin{aligned} \frac{\partial f_t(z)}{\partial t} &= (T^{-1})'(k_t(z))\frac{\partial k_t(T(z))}{\partial t}; \\ f_t'(z) &= (T^{-1})'(k_t(z))k_t'(T(z))T'(z). \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial f_t(z)}{\partial t} &= -f_t'(z)\frac{2}{T(z)-h(t)}\frac{1}{T'(z)} \\ &= -f_t'(z)\frac{2}{T(z)-h(t)}\frac{(1-z)^2}{2i} \\ &= (1-z)^2 f_t'(z)\frac{1}{\frac{z+1}{1-z}+ih(t)}, \qquad (z \in \mathbb{D}). \end{aligned}$$

Chordal Loewner Theory.

Notice that the function $p(z,t) := \frac{1}{\frac{z+1}{1-z}+ih(t)}$ has non-negative real part and we obtain that

$$\frac{\partial f_t(z)}{\partial t} = (1-z)^2 f_t'(z) p(z,t), \qquad (z \in \mathbb{D}). \tag{3.1}$$

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Again, the study of this PDE passes previously through the Initial Value Problem

$$\frac{dw}{dt} = -(1-w)^2 p(w,t), \quad w(s) = z, \tag{3.2}$$

when $s \geq 0$ and $z \in \mathbb{D}$. In this case, the solution does exist for $t \in [s, T_{z,s})$.

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$$\frac{dw}{dt} = -(1-w)^2 p(w,t), \quad w(s) = z, \tag{3.2}$$

when $s \geq 0$ and $z \in \mathbb{D}$. In this case, the solution does exist for $t \in [s, T_{z,s})$. Equations (3.1) and (3.2) are nowadays known as chordal partial or ordinary Loewner differential equation with the function h as driving term.
This kind of construction can be used to model evolutionary aspects of decreasing families of domains in the complex plane.

Chordal Loewner Theory.

Remark 1. Notice that in this case, we have a family of domains Ω_t such that $\Omega_t \subseteq \Omega_s$ whenever $s < t$.
This fact is shown in the sign "$-$" in equation

$$\frac{dw}{dt} = -(1-w)^2 p(w,t), \quad w(s) = z. \tag{3.3}$$

Equation

$$\frac{dw}{dt} = (1-w)^2 p(w,t), \quad w(s) = z. \tag{3.4}$$

has a solution for all $t \in [s, +\infty)$. Truncating adequately the function p, one can deduce the existence and properties of the solution of the initial value problem (3.3) from the solution of the initial value problem (3.4).

Chordal Loewner Theory.

Remark 2. Observe that in the radial case we are normalizing in an inner point of the reference domain and now we are normalizing at the point ∞ which belongs to the boundary of the upper half-plane or normalizing at the point 1 after rewriting the equation in the unit disk.
This is the reason the factor $(1-w)^2$ appears in

$$\frac{dw}{dt} = -(1-w)^2 p(w,t), \quad w(s) = z.$$

Chordal Loewner Theory.

In 2000 Schramm had the simple but very effective idea of replacing the function h in

$$\frac{\partial k_t(w)}{\partial t} = -k_t'(w)\frac{2}{w-h(t)}, \qquad (w \in \mathbb{H}).$$

by a Brownian motion and used the resulting chordal Loewner equation to understand critical processes in two dimensions, relating probability theory to complex analysis in a completely novel way.
Nowadays this equation is known as SLE (*stochastic Loewner equation*).
The book by Lawler is a good place to deep in this stochastic approach.

Semigroups of analytic functions.

Another mathematical construction closely related to Loewner theory is one-parametric semigroups of analytic functions.

This construction provides a set of important non-trivial examples for Operator Theory.

Semigroups of analytic functions.

Another mathematical construction closely related to Loewner theory is one-parametric semigroups of analytic functions.

Definition

A *(one-parametric) semigroup of analytic functions* in $\mathbb{D}$ is a family $(\phi_t)_{t\geq 0} \subset H(\mathbb{D},\mathbb{D})$ such that

- S1 $\phi_0 = \mathrm{id}_{\mathbb{D}}$;
- S2 $\phi_{t+s} = \phi_t \circ \phi_s$;
- S3 $\phi_t(z) \to z$ as $t \to +0$ for any $z \in \mathbb{D}$.

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These semigroups are a classical subject of study, as flows of continuous dynamical systems.

Semigroups also appear in connection with the theory of Galton-Watson processes (branching processes).

Furthermore, they provide the theory of continuous semigroups of operators (namely, between spaces of analytic functions) with an important factory of examples.

Semigroups of analytic functions.

Theorem

For any such semigroup (ϕ_t) *there exists a holomorphic function* $G : \mathbb{D} \to \mathbb{C}$ *such that*

$$G(z) = \lim_{t \to +0} \frac{\phi_t(z) - z}{t}, \qquad \frac{d\phi_t(z)}{dt} = G(\phi_t(z)).$$

G is known as the *infinitesimal generator* of the semigroup.

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Theorem (E. Berkson, H. Porta, 1978)

G is an infinitesimal generator iff $G(z) = (\tau - z)(1 - \bar{\tau} z)p(z)$, where $\tau \in \overline{\mathbb{D}}$ and $p : \mathbb{D} \to \mathbb{C}$ holomorphic and $\operatorname{Re} p \geq 0$.

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- If $\tau = 0$, then $G(z) = -zp(z)$, and the equation for $w(t) = \phi_t(z)$ is

$$dw/dt = -wp(w),$$

which is the autonomous version of the radial Loewner ODE;

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which is the autonomous version of the radial Loewner ODE;
- If $\tau = 1$, then $G(z) = (1 - z)^2 p(z)$, and
$$dw/dt = (1 - w)^2 p(w),$$
which resembles the autonomous chordal Loewner ODE.

Unified approach.

Our first contribution (with Bracci, Díaz-Madrigal, and Gumenyuk) to Loewner theory was to find a unified approach to the three settings we have mentioned.
Our motivation: find a general construction which contains, as particular cases:

- radial Loewner evolution,
- chordal Loewner evolution,
- semigroups of analytic functions.

Three basic notions must be considered

- Evolution families $(\varphi_{s,t})$.
- Herglotz vector fields $G(w,t)$.
- Loewner chains (f_t).

In the literature there is not an intrinsic definition of EF or “chordal Loewner chain” (as in the radial case Pommerenke did).

What should an evolution family be?

There have been several attempts to give a definition of EF:
Ω a simply connected domain ($\mathbb{D}$, $\mathbb{H}$, ...)
$\varphi_{s,t} : \Omega \to \Omega$, holomorphic (in fact, univalent), $0 \leq s \leq t < +\infty$:

- **EF1** $\varphi_{s,s} = \mathrm{id}_\Omega$,
- **EF2** $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$, whenever $0 \leq s \leq u \leq t < +\infty$,
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EF3 ?

EF3: Radial case
$\varphi_{s,t}(0) = 0$ and
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EF3: Chordal case
$\varphi_{s,t}(z) =$
$z - \frac{t-s}{z} + o(1/z)$
(hydrodynamic normalization at ∞).

EF3: Semigroup case
$\phi_t(z) \to z$ as $t \to +0$
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In this case, $\varphi_{s,t} := \phi_{t-s}$.

Problem

Assume $(\varphi_{s,t})$ satisfies EF1, EF2, and EF3.
Does $(\varphi_{s,t})$ come from a Loewner ODE?

What should an evolution family be?

Definition

A family $(\varphi_{s,t})$, $0 \leq s \leq t < +\infty$, of holomorphic self-maps of $\mathbb{D}$ is a *(generalized) evolution family of order* $d \in [1, +\infty]$ if

EF1 $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$,

EF2 $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $0 \leq s \leq u \leq t < +\infty$,

EF3 for any $z \in \mathbb{D}$ and $T > 0$ there is $k_{z,T} \in L^d([0,T], \mathbb{R})$ s. t.

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi)d\xi, \quad 0 \leq s \leq u \leq t \leq T.$$

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When $d = \infty$

EF3 for any $z \in \mathbb{D}$ and $T > 0$ there is a constant $c_{z,T}$ s. t.

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \le c_{z,T}(t-u), \quad 0 \le s \le u \le t \le T.$$

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In all the cases mentioned before, $(\varphi_{s,t})$ satisfies this definition with $d = \infty$.

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$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \le \int_u^t k_{z,T}(\xi) d\xi, \quad 0 \le s \le u \le t \le T.$$

Theorem

$\varphi_{s,t}$ is univalent.

Herglotz vector fields.

Definition

A (generalized) Herglotz function of order $d \in [1, +\infty]$ is a function $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ satisfying

HF1 For all $z \in \mathbb{D}$, the function $t \in [0, +\infty) \mapsto p(z, t) \in \mathbb{C}$ belongs to $L^d_{loc}([0, +\infty), \mathbb{C})$;

HF2 $p(\cdot, t)$ is holomorphic in $\mathbb{D}$ for a.e. $t \geq 0$;

HF3 For all z and for all t, we have $\operatorname{Re} p(z, t) \geq 0$.

Main result

Theorem

There is an essentially 1-to-1 correspondence between evolution families $(\varphi_{s,t})$ *of order* d *and couples* (τ, p) *where* $\tau : [0, +\infty) \to \overline{\mathbb{D}}$ *is a measurable function and* p *is a Herglotz function of order* d*.*

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Theorem

There is an essentially 1-to-1 correspondence between evolution families $(\varphi_{s,t})$ of order d and couples (τ, p) where $\tau : [0, +\infty) \to \overline{\mathbb{D}}$ is a measurable function and p is a Herglotz function of order d.

That is, given $(\varphi_{s,t})$ there is (τ, p) such that for all $s \geq 0$ and $z \in \mathbb{D}$, the map $t \in [s, +\infty) \mapsto w(t) := \varphi_{s,t}(z)$ is the solution for the Loewner ODE

$$\frac{dw}{dt} = (w - \tau(t))(\overline{\tau(t)}w - 1)p(w, t), \quad w(s) = z. \qquad (5.1)$$

Conversely, given (τ, p), solving (5.1), $\varphi_{s,t}(z) = w(t)$ is an EF.

Definition

$G(w, t) := (w - \tau(t))(\overline{\tau(t)}w - 1)p(w, t)$ is a *(generalized) Herglotz vector field.*

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(Generalized) Loewner chains

Definition

A family $(f_t)_{0\leq t<+\infty}$ of holomorphic maps of the unit disc is called a *(generalized) Loewner chain of order* $d \in [1,+\infty]$ if

LC1 each function $f_t : \mathbb{D} \to \mathbb{C}$ is univalent,

LC2 $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leq s < t < +\infty$,

LC3 for any compact set $K \subset \mathbb{D}$ and all $T > 0$ there exists $k_{K,T} \in L^d([0,T],\mathbb{R})$ such that

$$\sup_{z\in K} |f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\xi)d\xi, \quad 0 \leq s \leq t \leq T.$$

(Generalized) Loewner chains

Theorem

- *For any Loewner chain (f_t) of order d, the formula*

$$\varphi_{s,t} = f_t^{-1} \circ f_s$$

  defines an evolution family of the same order d.
- *For any evolution family $(\varphi_{s,t})$ of order d, there exists a unique Loewner chain (f_t) of the same order d s. t.:*
  (a) $\varphi_{s,t} = f_t^{-1} \circ f_s$ *for any* $t \geq s \geq 0$;
  (b) $f_0 \in \mathcal{S}$, *and*
  (c) $\Omega := \cup_{t \geq 0} f_t(\mathbb{D}) = \{z : |z| < R\}$, $R \in (0, +\infty]$.

(Generalized) Loewner chains

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- M.D. C., S. Díaz-Madrigal, and P. Gumenyuk, *Loewner Chains and Evolution Families*. Rev. Mat. Iberoam. **26** (2010), 975-1012.

Scheme of correspondence

Loewner chains (f_t) $\xleftrightarrow{\varphi_{s,t}=f_t^{-1}\circ f_s}$ Evolution families $(\varphi_{s,t})$

Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = -G(z,t)\frac{\partial f_t(z)}{\partial z}$$

Loewner ODE

$$\frac{dw}{dt} = G(w,t),\ w(s) = z$$

$$\varphi_{s,t}(z) = w(t)$$

Herglotz vector fields $G(w,t)$

where $G(w,t) = (w-\tau(t))(\overline{\tau(t)}w-1)p(w,t)$.

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Take (τ, p) and $G(w,t) = (w - \tau(t))(\overline{\tau(t)}w - 1)p(w,t)$.

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- If $G(z,t) = G(z)$ does not depend on t, then the evolution family we obtain is $\varphi_{s,t} = \phi_{t-s}$, where (ϕ_t) is the semigroup generated by G.

Open problems

Now, I present a list of open problems I am interested in.

Problem: New univalence criteria

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Problem

Find criteria to ensure that a solution of the general (or chordal) Loewner PDE is univalent.

Obtain new univalence criteria and characterization of starlike functions from a boundary point.

Problem: Extensions of Loewner chain beyond $\mathbb{D}$

An interesting problem is to analyze when the solution of Loewner's equations admit quasiconformal extensions beyond the unit disk.
This is a fundamental tool to apply Loewner theory to fluid dynamics (see the book by Gustafsson and Vasiliev).

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In 1972, J. Becker proved that if p is a Herglotz function and there are numbers $0 < a < b$ such that

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Problem

Find sufficient and necessary conditions on a Herglotz vector field that ensure that the associated Loewner chain and/or the evolution family are "regular" beyond the unit disk.

Problem: Chordal Loewner chain

A Loewner chain is chordal when its associated Herglotz vector field has a factorization of the type $G(w,t) = (w-1)^2 p(w,t)$.
In the paper
-M. D. C., S. Díaz-Madrigal, and P. Gumenyuk, *Geometry behind chordal Loewner chains*, Complex Anal. Oper. Theory **4** (2010), 541–587.
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Problem

Give an intrinsic characterization of chordal Loewner chains.[a]

[a]This problem for evolution families was solved (all the functions $\varphi_{s,t}$ must share the Denjoy-Wolff point and it must be at the boundary of $\mathbb{D}$) in
-F. Bracci, M.D. Contreras, and S. Díaz-Madrigal, *Evolution Families and the Loewner Equation I: the unit disk*, J. Reine Angew. Math.

Problem: Stochastic approach

Without doubt, the success of the chordal equation is due to the many applications of its stochastic version.
As we have seen, the starting point of Schramm development was to replace the driving term h by a Brownian motion in

$$\frac{\partial k_t(w)}{\partial t} = -k_t'(w)\frac{2}{w-h(t)}, \qquad (w \in \mathbb{H}).$$

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With the new general Loewner theory, one can introduce such ideas, for example, replacing τ in the Herglotz vector field.

Problem

Analyze the evolution family and the Loewner chain with a vector field

$$G(z,t) = (z-\tau(t))(\overline{\tau(t)}z-1)p(z,t),$$

where $\tau : [0,\infty) \to \overline{\mathbb{D}}$ *is a Brownian motion. One can begin by the case* $p \equiv 1$.

Problem: Loewner Theory in multiply connected domains

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Then

$$\varphi_{s,t}(z) = e^{(\lambda(s)-\lambda(t))i} z \quad \text{and} \quad f_s(D) = f_t(D).$$

Everything would be trivial!

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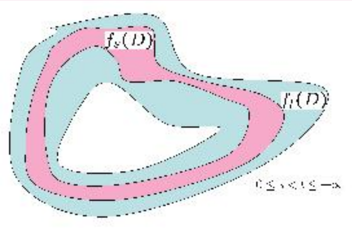
How can we study an evolution of domains like the following?



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In the double connected case, this problem has been deal with by

Slit case: Komatu (1943, 1950), Goluzin (1951), Li En Pir (1953), Lebedev (1955), Kuvaev and Kufarev (1955).

and more recently

- M. D. C., S. Díaz-Madrigal, and P. Gumenyuk, *Loewner Theory in annulus I: evolution families and differential equations.* Preprint 2010. Availiable on arXiv:1011.4253
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Problem

Extend Loewner Theory to multiply connected domains with $n > 2$.