Loewner’s Theory: a parametric method to tackle problems in Complex Analysis

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Schedule of the lectures.

- The Bieberbach Conjecture
  1. The area theorem and the case $n = 2$ in Bieberbach Conjecture
  2. Some applications: distortion theorems

- Radial Loewner Theory
  1. First steps in Loewner theory
  2. From Loewner chains to Loewner PDE and the case $n = 3$ in Bieberbach conjecture
  3. From Loewner PDE to Loewner chains: evolution families
  4. Some applications: Univalence criteria

- Chordal Loewner Theory

- Semigroups of analytic functions

- General Loewner Theory

- Some open problems
First steps in Loewner theory

Definition

A family of holomorphic functions in $\mathbb{D}$, $(f_t)_{t \geq 0}$, is said to be a (radial) Loewner chain, if

- **LC1** all $f_t$’s are univalent in $\mathbb{D}$;
- **LC2** $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leq s \leq t$;
- **LC3** for any $t \geq 0$, $f_t(z) = e^t z + a_2(t)z^2 + \ldots$, i.e., $e^{-t}f_t \in S$.

Theorem

For any $f \in S$, there exists a Loewner chain $(f_t)$ such that $f = f_0$.

Theorem

Let $(f_t)$ be a Loewner chain. Then there exists a Herglotz function $p$ such that $\frac{\partial f_t(z)}{\partial t} = zf_t'(z)p(z, t)$. 
First steps in Loewner theory

Definition

We say that $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is a Herglotz function if

1. $z \mapsto p(z, t)$ is analytic for all $t$;
2. $t \mapsto p(z, t)$ is measurable for all $z$;
3. $p(0, t) = 1$ for all $t$;
4. $\text{Re} \ p(z, t) > 0$ for all $z$ and all $t$. 
From Loewner PDE to Loewner chains

In the section we prove that any Loewner PDE has a solution which is a Loewner chain.

We have to solve firstly a certain ODE.
From Loewner PDE to Loewner chains

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We have to solve firstly a certain ODE.

We apply this result to get univalence criteria for analytic maps.
From Loewner PDE to Loewner chains

**Theorem**

Let \( p : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C} \) be a Herglotz function. Then, for any \( z \in \mathbb{D} \) and any \( s \geq 0 \) the initial value problem

\[
\frac{dw}{dt} = -wp(w, t) \quad \text{for a.e. } t \in [s, +\infty), \quad w(s) = z
\]

(Loewner ODE)

has a unique solution \( w(t) \) in \( t \in [s, +\infty) \).
From Loewner PDE to Loewner chains

**Theorem**

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has a unique solution \( w(t) \) in \( t \in [s, +\infty) \). Write \( \varphi_{s,t}(z) := w(t) \).

Then \( \varphi_{s,t} \) is a univalent for all \( 0 \leq s \leq t < +\infty \) and

1. **EF1** \( \varphi_{s,s} = \text{id}_{\mathbb{D}} \),
2. **EF2** \( \varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u} \) whenever \( 0 \leq s \leq u \leq t < +\infty \),
3. **EF3** \( \varphi_{s,t}(0) = 0 \) and \( \varphi'_{s,t}(0) = e^{s-t} \).

**Definition**

A family of analytic self-maps of the unit disk \((\varphi_{s,t})_{0 \leq s \leq t < +\infty}\) satisfying EF1, EF2, and EF3 is called an evolution family.
From Loewner PDE to Loewner chains

Proof.
We define $w_0(z, t) \equiv 0$ and, recursively,

$$w_{n+1}(z, t) = z \exp \left[ - \int_s^t p(w_n(z, \tau), \tau) d\tau \right] \quad n = 0, 1, 2, 3...$$

for $t \in [s, +\infty)$ and for all $z \in \mathbb{D}$. Notice that $|w_n(z, t)| < 1$ for all $n$ and the function $z \mapsto w_n(z, t)$ is analytic for all $t$. 

Schwarz's lemma implies that $|w_n(z, t)| < |z|$. 

Proof.
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Schwarz’s lemma implies that $|w_n(z, t)| < |z|$. 
**Proof.** Using again the representation formula

\[ p(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \]

we deduce that

\[ |p'(z, \tau)| \leq 2(1 - |z|)^{-2}. \]
From Loewner PDE to Loewner chains

**Proof.** Using again the representation formula
\[ p(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \]
we deduce that
\[ |p'(z, \tau)| \leq 2(1 - |z|)^{-2}. \]
Moreover \(|e^{-a} - e^{-b}| \leq |a - b|\) whenever \(\text{Re } a, \text{Re } b \geq 0\). Thus
\[
|w_{n+1}(z, t) - w_n(z, t)| = \\
= |z| \left| \exp \left[ - \int_s^t p(w_n(z, \tau), \tau) d\tau \right] - \exp \left[ - \int_s^t p(w_{n-1}(z, \tau), \tau) d\tau \right] \right| \\
\leq \int_s^t |p(w_n(z, \tau), \tau) - p(w_{n-1}(z, \tau), \tau)| d\tau \\
= \int_s^t \left| \int_{w_n(z, \tau)}^{w_n(z, \tau)} p' (\xi, \tau) d\xi \right| d\tau \\
\leq \int_s^t \int_{w_n(z, \tau)}^{w_{n-1}(z, \tau)} \left| \frac{2}{(1 - |\xi|)^2} \right| d\xi d\tau \\
\leq \int_s^t \int_{w_n(z, \tau)}^{w_{n-1}(z, \tau)} \left| \frac{2}{(1 - |z|)^2} \right| d\xi d\tau \leq \frac{2}{(1 - |z|)^2} \int_s^t \left| w_n(z, \tau) - w_{n-1}(z, \tau) \right| d\tau.
\]
From Loewner PDE to Loewner chains

Proof.

\[ |w_{n+1}(z, t) - w_n(z, t)| \leq \frac{2}{(1 - |z|)^2} \int_s^t |w_n(z, \tau) - w_{n-1}(z, \tau)| \, d\tau. \]

By induction we have that

\[ |w_{n+1}(z, t) - w_n(z, t)| \leq \frac{2^n(t - s)^n}{(1 - |z|)^{2n}n!} \quad (z \in \mathbb{D}, \ n = 0, 1, 2, 3, \ldots). \]

We conclude that \( \lim_n w_n(z, t) \) exists uniformly in \( |z| \leq r, \ s \leq t \leq T \) for every \( r < 1 \) and \( T \).
From Loewner PDE to Loewner chains

Proof.

\[ |w_{n+1}(z, t) - w_n(z, t)| \leq \frac{2}{(1 - |z|)^2} \int_s^t |w_n(z, \tau) - w_{n-1}(z, \tau)| \, d\tau. \]

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We conclude that \( \lim_n w_n(z, t) \) exists uniformly in \( |z| \leq r, \ s \leq t \leq T \) for every \( r < 1 \) and \( T \).

Denoting the limit by \( \varphi_{s,t}(z) \), we obtain an analytic function in \( z \) and by the Lebesgue’s bounded convergence theorem we obtain that

\[ \varphi_{s,t}(z) = z \exp \left[ - \int_s^t p(\varphi_{s,\tau}(z), \tau) \, d\tau \right]. \]

From this equation, it is easy to deduce the properties of \( \varphi_{s,t} \).
Corollary

Take $p$ and $(\varphi_{s,t})$ as in the above theorem. Then

$$f_s(z) = \lim_{t \to +\infty} e^t \varphi_{s,t}(z)$$

exists uniformly on compacta of $\mathbb{D}$, $(f_t)$ is a Loewner chain satisfying $f_s = f_t \circ \varphi_{s,t}$ and

$$\frac{\partial f_t(z)}{\partial t} = zf_t'(z)p(z, t).$$
From Loewner PDE to Loewner chains

Proof.
We just obtain the existence of the limit
\( f_s(z) = \lim_{t \to +\infty} e^t \varphi_{s,t}(z) \) and \( f_s = f_t \circ \varphi_{s,t} \). The rest of the proof is easy.

By the proof of the above theorem, we have that

\[
\varphi_{s,t}(z) = z \exp \left[ - \int_s^t p(\varphi_{s,\tau}(z), \tau) d\tau \right].
\]

Therefore

\[
e^{t-s} \varphi_{s,t}(z) = z \exp \left[ \int_s^t (1 - p(\varphi_{s,\tau}(z), \tau)) d\tau \right]
\]

and this function belongs to \( S \). By Koebe distortion theorem, we have

\[
|\varphi_{s,t}(z)| \leq \frac{|z|}{(1 - |z|)^2} e^{s-t}.
\]
Proof.
Fix \( r < 1 \). If \(|z| < r\) then \(|\varphi_{s,t}(z)| \leq r\) and

\[
|1 - p(\varphi_{s,\tau}(z), \tau)| = \left| \int_0^{\varphi_{s,\tau}(z)} p'(\xi, \tau) d\xi \right| \leq \frac{2}{(1 - r)^2} |\varphi_{s,\tau}(z)| \leq \frac{2}{(1 - r)^4} e^{s-\tau}.
\]

Thus, if \(|z| \leq r\) and \(t, t' \geq s\), after some computations, we have

\[
|e^{t-s}\varphi_{s,t}(z) - e^{t'-s}\varphi_{s,t'}(z)| = \ldots =
\]

\[
= |e^{t-s}\varphi_{s,t}(z)| \left| 1 - \exp \left[ \int_t^{t'} (1 - p(\varphi_{s,\tau}(z), \tau)) d\tau \right] \right|
\]

\[
\leq \frac{|z|}{(1 - |z|)^2} \left| 1 - \exp \left[ \int_t^{t'} (1 - p(\varphi_{s,\tau}(z), \tau)) d\tau \right] \right|.
\]

Thus the family \((e^{t-s}\varphi_{s,t})_{t \geq s}\) is Cauchy in the compact set \(r\overline{\mathbb{D}}\).
Proof.
Thus the limit

$$f_s(z) = \lim_{t \to +\infty} e^t \varphi_{s,t}(z)$$

is well defined. Moreover, if $s < t$,

$$f_s(z) = \lim_{\tau \to +\infty} e^{\tau} \varphi_{s,\tau}(z) = \lim_{\tau \to +\infty} e^{\tau} \varphi_{t,\tau} \circ \varphi_{s,t}(z) = f_t(\varphi_{s,t}(z)).$$
So far, we have applied Loewner theory just in one direction: if \((f_t)\) is a Loewner chain then it is the solution of a certain PDE

\[
\frac{\partial f_t(z)}{\partial t} = zf'_t(z)p(z, t).
\]

Moreover, we have learned that such PDE has a solution which is a Loewner chain.
Univalence

So far, we have applied Loewner theory just in one direction: if \((f_t)\) is a Loewner chain then it is the solution of a certain PDE

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Moreover, we have learned that such PDE has a solution which is a Loewner chain. But it can have other solutions: take \(h\) an entire function and write \(g_t := h \circ f_t\). Then

\[
\frac{\partial g_t(z)}{\partial t} = h'(f_t(z)) \frac{\partial f_t(z)}{\partial t} = h'(f_t(z))zf_t'(z)p(z, t) = zg_t'(z)p(z, t).
\]

The functions \(g_t\) are not univalent unless \(h(z) = az + b\).
So far, we have applied Loewner theory just in one direction: if \((f_t)\) is a Loewner chain then it is the solution of a certain PDE

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\]

The functions \(g_t\) are not univalent unless \(h(z) = az + b\).

The goal of this section is to characterize Loewner chains between all the solution of the above PDE and, as a by product of such result, we plan to get univalence criteria.
Univalence

**Theorem (Univalence Theorem)**

Let $0 < r < 1$. Let $(f_t)$ be a family of analytic maps in $r \mathbb{D}$ s.t.

1. $f_t(0) = 0$ and $f_t'(0) = e^t$;
2. For each $z \in r \mathbb{D}$, the map $t \mapsto f_t(z)$ is abs. continuous;
3. There is $K$ such that $|f_t(z)| \leq Ke^t$ for all $|z| < r$, and $t \geq 0$;
4. There exists a Herglotz function $p : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ s.t.

$$\frac{\partial f_t(z)}{\partial t} = zf_t'(z)p(z, t) \quad \text{for all } |z| < r, \text{ and } t \geq 0.$$ 

Then the function $f_t$ has an analytic continuation to $\mathbb{D}$ and $(f_t)$ is a Loewner chain.
Univalence

Proof. Take \((\varphi_{s,t})\) the evolution family associated with \(p\). Fix \(z \in rD\).

Using again Vitali’s theorem, \(t \mapsto k(t) := f_t(\varphi_{s,t}(z))\) is absolutely continuous in every interval of the form \([s, T]\).

Then

\[
    k'(t) = f'_t(\varphi_{s,t}(z)) \frac{\partial}{\partial t}(\varphi_{s,t}(z)) + \frac{\partial f_t}{\partial t}(\varphi_{s,t}(z))
\]

\[
    = -f'_t(\varphi_{s,t}(z))\varphi_{s,t}(z)p(\varphi_{s,t}(z), t)
    + \varphi_{s,t}(z)f'_t(\varphi_{s,t}(z))p(\varphi_{s,t}(z), t) = 0.
\]

But \(k(s) = f_s(z)\). Then \(f_s(z) = k(s) = k(t) = f_t \circ \varphi_{s,t}(z)\) whenever \(|z| < r\).

...
Univalence

Proof. ... By Schwarz’s lemma (applied twice), if $h : \mathbb{D} \to \mathbb{D}$ is analytic and $h(0) = h'(0) = 0$, then $|h(w)| \leq |w|^2$ for all $w \in \mathbb{D}$. 
Univalence

Proof. ... By Schwarz’s lemma (applied twice), if $h : \mathbb{D} \to \mathbb{D}$ is analytic and $h(0) = h'(0) = 0$, then $|h(w)| \leq |w|^2$ for all $w \in \mathbb{D}$.

Write $h(w) = \frac{e^{-t}f_t(rw) - rw}{K+1}$. By hypothesis, we have $h$ is an analytic self-map of $\mathbb{D}$ such that $h(0) = h'(0) = 0$. Thus, taking $z = rw$ we have

$$|e^{-t}f_t(z) - z| \leq (K + 1)r^{-2}|z|^2.$$
**Univalence**

**Proof.** By Schwarz’s lemma (applied twice), if $h : \mathbb{D} \rightarrow \mathbb{D}$ is analytic and $h(0) = h'(0) = 0$, then $|h(w)| \leq |w|^2$ for all $w \in \mathbb{D}$.

Write $h(w) = \frac{e^{-tf_t(rw)}-rw}{K+1}$. By hypothesis, we have $h$ is an analytic self-map of $\mathbb{D}$ such that $h(0) = h'(0) = 0$. Thus, taking $z = rw$ we have

$$|e^{-tf_t(z)} - z| \leq (K + 1)r^{-2}|z|^2.$$ 

Hence, using this inequality and applying Koebe distortion theorem to the function $e^{s-t}\varphi_{s,t}$, we have

$$|f_s(z) - e^t\varphi_{s,t}(z)| = e^t|e^{-tf_t(\varphi_{s,t}(z))} - \varphi_{s,t}(z)| \\ \leq e^t(K + 1)r^{-2}|\varphi_{s,t}(z)|^2 \\ \leq e^t(K + 1)r^{-2}|z|^2(1 - |z|)^{-4}e^{2s-2t} \\ \leq (K + 1)(1 - r)^{-4}e^{2s-t}.$$
Proof. ... By Schwarz’s lemma (applied twice), if \( h : \mathbb{D} \rightarrow \mathbb{D} \) is analytic and \( h(0) = h'(0) = 0 \), then \(|h(w)| \leq |w|^2\) for all \( w \in \mathbb{D} \).

Write \( h(w) = e^{-t}f_t(rw) - rw \). By hypothesis, we have \( h \) is an analytic self-map of \( \mathbb{D} \) such that \( h(0) = h'(0) = 0 \). Thus, taking \( z = rw \) we have

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|e^{-t}f_t(z) - z| \leq (K + 1)r^{-2}|z|^2.
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Hence, using this inequality and applying Koebe distortion theorem to the function \( e^{s-t}\varphi_{s,t} \), we have

\[
|f_s(z) - e^t\varphi_{s,t}(z)| = e^t|e^{-t}f_t(\varphi_{s,t}(z)) - \varphi_{s,t}(z)|
\]
\[
\leq e^t(K + 1)r^{-2}|\varphi_{s,t}(z)|^2
\]
\[
\leq e^t(K + 1)r^{-2}|z|^2(1 - |z|)^{-4}e^{2s-2t}
\]
\[
\leq (K + 1)(1 - r)^{-4}e^{2s-t}.
\]

Therefore, \( e^t\varphi_{s,t}(z) \rightarrow f_s(z) \) as \( t \rightarrow \infty \) for \( |z| < r \).
Proof. ... By Schwarz’s lemma (applied twice), if $h : \mathbb{D} \rightarrow \mathbb{D}$ is analytic and $h(0) = h'(0) = 0$, then $|h(w)| \leq |w|^2$ for all $w \in \mathbb{D}$.

Write $h(w) = \frac{e^{-t}f_t(rw) - rw}{K+1}$. By hypothesis, we have $h$ is an analytic self-map of $\mathbb{D}$ such that $h(0) = h'(0) = 0$. Thus, taking $z = rw$ we have

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Hence, using this inequality and applying Koebe distortion theorem to the function $e^{s-t}\varphi_{s,t}$, we have

$$|f_s(z) - e^t\varphi_{s,t}(z)| = e^t|e^{-t}f_t(\varphi_{s,t}(z)) - \varphi_{s,t}(z)|$$

$$\leq e^t(K + 1)r^{-2}|\varphi_{s,t}(z)|^2$$

$$\leq e^t(K + 1)r^{-2}|z|^2(1 - |z|)^{-4}e^{2s-2t}$$

$$\leq (K + 1)(1 - r)^{-4}e^{2s-t}.$$ 

Therefore, $e^t\varphi_{s,t}(z) \rightarrow f_s(z)$ as $t \rightarrow \infty$ for $|z| < r$.

This implies that $f_s$ is nothing but the restriction to $r\mathbb{D}$ of the function built in the last corollary.
Starlike functions

A function $f$ is called spirallike of type $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ if it is univalent in $\mathbb{D}$ and $\exp(-e^{i\alpha} t)f(\mathbb{D}) \subseteq f(\mathbb{D})$ for all $t \geq 0$. Starlike functions are the special case $\alpha = 0$.

**Corollary (Spacek, 1933, Robertson, 1961)**

The function $f(z) = z + \ldots$ analytic in $\mathbb{D}$ is spirallike of type $\alpha$ if and only if

$$\text{Re} \left[ e^{i\alpha} z \frac{f'(z)}{f(z)} \right] > 0 \quad \text{for all } z \in \mathbb{D}.$$
Corollary (Spacek, 1933, Robertson, 1961)

The function \( f(z) = z + \ldots \) analytic in \( \mathbb{D} \) is starlike if and only if

\[
\text{Re} \left[ z \frac{f'(z)}{f(z)} \right] > 0 \quad \text{for all } z \in \mathbb{D}.
\]

**Proof.** Define \( f_t(z) = e^t f(z) = e^t z + \ldots \) for all \( z \in \mathbb{D} \) and \( t \geq 0 \). Since

\[
\frac{\partial f_t(z)}{\partial t} = \frac{e^t f(z)}{zf_t'(z)} = \frac{f(z)}{zf'(z)},
\]

defining \( p(z, t) = \frac{f(z)}{zf'(z)} \) it is a non-negative function for which

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\frac{\partial f_t(z)}{\partial t} = zf_t'(z)p(z, t).
\]
Starlike functions

**Corollary (Spacek, 1933, Robertson, 1961)**

The function $f(z) = z + \ldots$ analytic in $\mathbb{D}$ is starlike if and only if

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**Proof.** Define $f_t(z) = e^t f(z) = e^t z + \ldots$ for all $z \in \mathbb{D}$ and $t \geq 0$. Since

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defining $p(z, t) = \frac{f(z)}{zf'(z)}$ it is a non-negative function for which

$$\frac{\partial f_t(z)}{\partial t} = zf_t'(z)p(z, t).$$

Clearly $(f_t)$ satisfies the hypothesis of Univalence Theorem and $f_t$ is a Loewner chain. This implies that $f = f_0$ is univalent and $f(\mathbb{D}) = f_0(\mathbb{D}) \subseteq f_t(\mathbb{D}) = e^tf(\mathbb{D})$, that is, $e^{-t}f(\mathbb{D}) \subseteq f(\mathbb{D})$ for all $t$. 
Corollary (Spacek, 1933, Robertson, 1961)

The function $f(z) = z + \ldots$ analytic in $\mathbb{D}$ is starlike if and only if

$$\text{Re} \left[ z \frac{f'(z)}{f(z)} \right] > 0 \quad \text{for all } z \in \mathbb{D}.$$

**Proof.** Define $f_t(z) = e^t f(z) = e^t z + \ldots$ for all $z \in \mathbb{D}$ and $t \geq 0$. Since

$$\frac{\partial f_t(z)}{\partial t} / (zf_t'(z)) = \frac{e^t f(z)}{ze^t f'(z)} = \frac{f(z)}{zf'(z)},$$

defining $p(z, t) = \frac{f(z)}{zf'(z)}$ it is a non-negative function for which

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Univalence criteria

From distortion theorem, one can prove that any \( f \in S \) satisfies

\[
(1 - |z|^2) \left| z \frac{f''(z)}{f'(z)} \right| < 6 \quad (z \in \mathbb{D}).
\]

The converse is not true.
Take \( f(z) = \exp(\lambda z) \), with \( \lambda > \pi \). \( f \) is not univalent in \( \mathbb{D} \) and

\[
(1 - |z|^2) \left| z \frac{f''(z)}{f'(z)} \right| \leq \frac{2\sqrt{3} \lambda}{9}, \quad \frac{2\sqrt{3} \pi}{9} \approx 1.209...
\]

But

**Corollary (Becker, 1972)**

*Let \( f \) be analytic in \( \mathbb{D} \) with \( f'(0) \neq 0 \). If

\[
(1 - |z|^2) \left| z \frac{f''(z)}{f'(z)} \right| < 1 \quad (z \in \mathbb{D})
\]

then \( f \) is univalent in \( \mathbb{D} \).*
Proof. We may assume that $f'(0) = 1$. Let

$$f_t(z) := f(e^{-t}z) + (e^t - e^{-t})zf'(e^{-t}z)$$

for all $z \in \mathbb{D}$ and $t \geq 0$.

Then

$$\frac{\partial f_t(z)}{\partial t} = e^t zf'(e^{-t}z) - (1 - e^{-2t})z^2 f''(e^{-t}z),$$

$$zf'_t(z) = e^t zf'(e^{-t}z) + (1 - e^{-2t})z^2 f''(e^{-t}z).$$
Univalence criteria

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\]
for all \( z \in \mathbb{D} \) and \( t \geq 0 \).

Then
\[
\frac{\partial f_t(z)}{\partial t} = e^t zf'(e^{-t}z) - (1 - e^{-2t})z^2 f''(e^{-t}z),
\]
and
\[
zf'_t(z) = e^t zf'(e^{-t}z) + (1 - e^{-2t})z^2 f''(e^{-t}z).
\]

Write \( p(z, t) := \frac{\partial f_t(z)}{\partial t} / (zf'_t(z)) \). Then
\[
\left| \frac{p(z, t) - 1}{p(z, t) + 1} \right| = \left| \frac{\partial f_t(z)}{\partial t} - zf'_t(z) \right| =
\]
\[
= (1 - e^{-2t}) \left| e^{-t} z \frac{f''(e^{-t}z)}{f'(e^{-t}z)} \right| \leq \frac{1 - e^{-2t}}{1 - e^{-2t}|z|^2} < 1.
\]
Proof. We may assume that $f'(0) = 1$. Let

$$f_t(z) := f(e^{-t}z) + (e^t - e^{-t})zf'(e^{-t}z)$$

for all $z \in \mathbb{D}$ and $t \geq 0$.

Then

$$\frac{\partial f_t(z)}{\partial t} = e^t z f'(e^{-t}z) - (1 - e^{-2t})z^2 f''(e^{-t}z),$$

$$zf_t'(z) = e^t z f'(e^{-t}z) + (1 - e^{-2t})z^2 f''(e^{-t}z).$$

Write $p(z, t) := \frac{\partial f_t(z)}{\partial t} / (zf_t'(z))$. Then

$$\left| \frac{p(z, t) - 1}{p(z, t) + 1} \right| = \left| \frac{\partial f_t(z)}{\partial t} - zf_t'(z) \right| = \left| \frac{\partial f_t(z)}{\partial t} + zf_t'(z) \right| = (1 - e^{-2t}) \left| e^{-t} z f''(e^{-t}z) f'(e^{-t}z) \right| \leq \frac{1 - e^{-2t}}{1 - e^{-2t}|z|^2} < 1.$$ 

The map $T(z) = \frac{z + 1}{1 - z}$ sends the unit disk into the right half-plane. But $p(z, t) = T\left( \frac{p(z,t) - 1}{p(z,t) + 1} \right)$. 
Proof. We may assume that $f'(0) = 1$. Let 

$$f_t(z) := f(e^{-t}z) + (e^t - e^{-t})zf'(e^{-t}z) \quad \text{for all } z \in \mathbb{D} \text{ and } t \geq 0.$$ 

Then 

$$\frac{\partial f_t(z)}{\partial t} = e^t zf'(e^{-t}z) - (1 - e^{-2t})z^2f''(e^{-t}z),$$

$$zf'_t(z) = e^t zf'(e^{-t}z) + (1 - e^{-2t})z^2f''(e^{-t}z).$$

Write $p(z, t) := \frac{\partial f_t(z)}{\partial t}/(zf'_t(z))$. Then 

$$\left| \frac{p(z, t) - 1}{p(z, t) + 1} \right| = \left| \frac{\frac{\partial f_t(z)}{\partial t} - zf'_t(z)}{\frac{\partial f_t(z)}{\partial t} + zf'_t(z)} \right| =$$

$$(1 - e^{-2t}) \left| e^{-t}z \frac{f''(e^{-t}z)}{f'(e^{-t}z)} \right| \leq \frac{1 - e^{-2t}}{1 - e^{-2t}|z|^2} < 1.$$ 

The map $T(z) = \frac{z+1}{1-z}$ sends the unit disk into the right half-plane. But $p(z, t) = T \left( \frac{p(z, t) - 1}{p(z, t) + 1} \right)$. Therefore $\text{Re } p(z, t) > 0$. ...
Proof. ...
Since \( \frac{\partial f_t(z)}{\partial t} = zf_t'(z)p(z, t) \), the family \((f_t)\) satisfies the hypothesis of Univalence Theorem.

Thus \((f_t)\) is a Loewner chain and, in particular, \(f = f_0\) is univalent.
Summing up, we have introduced three concepts: evolutions families, Herglotz functions and Loewner chains. There is a one to one correspondence between these notions:

Loewner chains \( (f_t) \) \( \leftrightarrow \) Evolution families \( (\varphi_{s,t}) \)

Loewner PDE

\[
\frac{\partial f_t(z)}{\partial t} = z p(z, t) f_t'(z)
\]

Herglotz function \( p(w, t) \)

Loewner ODE

\[
\frac{dw}{dt} = -z p(w, t), \quad w(s) = z
\]

\[
\varphi_{s,t}(z) := w(t)
\]
Summing up, we have introduced three concepts: evolutions families, Herglotz functions and Loewner chains. There is a one to one correspondence between these notions:

\[ \varphi_{s,t} = f_t^{-1} \circ f_s \]

Loewner chains \((f_t)\) \(\leftrightarrow\) Evolution families \((\varphi_{s,t})\)

\[ \frac{\partial f_t(z)}{\partial t} = zp(z, t)f_t'(z) \]

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\[ \frac{dw}{dt} = -zp(w, t), \; w(s) = z \]

Loewner ODE

\[ \varphi_{s,t}(z) := w(t) \]

Herglotz function \(p(w, t)\)

Loewner’s theory has been used to prove several deep results in various branches of mathematics, even apparently unrelated to complex analysis. So far, we have presented two of them but there are many other. See Duren and Pommerenke.
Chordal Loewner Theory.

Consider a Jordan arc \( \Gamma \) contained in \( \mathbb{H} = \{ w : \text{Im} w > 0 \} \) except for its endpoint \( a \in \mathbb{R} \) parameterized by
\[
\gamma : [0, T] \to \mathbb{H} \cup \{a\}.
\]
Write \( \Omega_t := \mathbb{H} \setminus \gamma([t, T]), t \geq 0 \).

\[
\mathbb{H} := \{w : \text{Im} w > 0\}
\]

\[
a = \gamma(T)
\]
Consider a Jordan arc $\Gamma$ contained in $\mathbb{H} = \{w : \text{Im } w > 0\}$ except for its endpoint $a \in \mathbb{R}$ parameterized by $\gamma : [0, T] \to \mathbb{H} \cup \{a\}$.

Write $\Omega_t := \mathbb{H} \setminus \gamma([t, T]), \ t \geq 0$.

Then there is a unique univalent and onto $k_t : \mathbb{H} \to \Omega_t$ such that

$$k_t(z) = z - \frac{c(t)}{z} + o(1/z^2), \quad c(t) \geq 0, \ z \to \infty.$$ 

Up to a reparametrization we may assume that $c(t) = T - t$. 

![Diagram](image)
Under above normalization, Kufarev, Sobolev and Sporysheva (1968) showed that $k_t$ satisfies the following partial differential equation:

$$ \frac{\partial k_t(w)}{\partial t} = -k'_t(w) \frac{2}{w - h(t)}, \quad (w \in \mathbb{H}), $$

where $h$ is a continuous real-valued function called “driving term”.
Chordal Loewner Theory.

To compare this PDE with the Loewner PDE, we should rewrite them in the same setting, for example the unit disk.
Chordal Loewner Theory.

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$$\frac{\partial f_t(z)}{\partial t} = (T^{-1})'(k_t(z)) \frac{\partial k_t(T(z))}{\partial t};$$
$$f'_t(z) = (T^{-1})'(k_t(z))k'_t(T(z))T'(z).$$
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$$\frac{\partial f_t(z)}{\partial t} = (T^{-1})'(k_t(z)) \frac{\partial k_t(T(z))}{\partial t};$$

$$f_t'(z) = (T^{-1})'(k_t(z))k_t'(T(z))T'(z).$$

Thus

$$\frac{\partial f_t(z)}{\partial t} = -f_t'(z) \frac{2}{T(z) - h(t)} \frac{1}{T'(z)}$$

$$= -f_t'(z) \frac{2}{T(z) - h(t)} \frac{(1 - z)^2}{2i}$$

$$= (1 - z)^2 f_t'(z) \frac{1}{\frac{z+1}{1-z} + ih(t)}, \quad (z \in \mathbb{D}).$$
Chordal Loewner Theory.

Notice that the function $p(z, t) := \frac{1}{\frac{z+1}{1-z} + ih(t)}$ has non-negative real part and we obtain that

$$\frac{\partial f_t(z)}{\partial t} = (1 - z)^2 f'_t(z) p(z, t), \quad (z \in \mathbb{D}). \quad (3.1)$$
Chordal Loewner Theory.

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Again, the study of this PDE passes previously through the Initial Value Problem

$$\frac{dw}{dt} = -(1 - w)^2 p(w, t), \quad w(s) = z, \quad (3.2)$$

when $s \geq 0$ and $z \in \mathbb{D}$. In this case, the solution does exist for $t \in [s, T_{z,s})$. 
Chordal Loewner Theory.

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Again, the study of this PDE passes previously through the Initial Value Problem

\[
\frac{dw}{dt} = -(1 - w)^2 p(w, t), \quad w(s) = z, \tag{3.2}
\]

when \( s \geq 0 \) and \( z \in \mathbb{D} \). In this case, the solution does exist for \( t \in [s, T_{z,s}) \). Equations (3.1) and (3.2) are nowadays known as chordal partial or ordinary Loewner differential equation with the function \( h \) as driving term. This kind of construction can be used to model evolutionary aspects of decreasing families of domains in the complex plane.
Remark 1. Notice that in this case, we have a family of domains $\Omega_t$ such that $\Omega_t \subseteq \Omega_s$ whenever $s < t$. This fact is shown in the sign “−” in equation

$$\frac{dw}{dt} = -(1 - w)^2 p(w, t), \quad w(s) = z. \quad (3.3)$$

Equation

$$\frac{dw}{dt} = (1 - w)^2 p(w, t), \quad w(s) = z. \quad (3.4)$$

has a solution for all $t \in [s, +\infty)$. Truncating adequately the function $p$, one can deduce the existence and properties of the solution of the initial value problem (3.3) from the solution of the initial value problem (3.4).
Remark 2. Observe that in the radial case we are normalizing in an inner point of the reference domain and now we are normalizing at the point $\infty$ which belongs to the boundary of the upper half-plane or normalizing at the point 1 after rewriting the equation in the unit disk. This is the reason the factor $(1 - w)^2$ appears in

$$
\frac{dw}{dt} = -(1 - w)^2 p(w, t), \quad w(s) = z.
$$
Chordal Loewner Theory.

In 2000 Schramm had the simple but very effective idea of replacing the function $h$ in

$$\frac{\partial k_t(w)}{\partial t} = -k'_t(w) \frac{2}{w - h(t)}, \quad (w \in \mathbb{H}).$$

by a Brownian motion and used the resulting chordal Loewner equation to understand critical processes in two dimensions, relating probability theory to complex analysis in a completely novel way. Nowadays this equation is known as SLE (stochastic Loewner equation).

The book by Lawler is a good place to deep in this stochastic approach.
Semigroups of analytic functions.

Another mathematical construction closely related to Loewner theory is one-parametric semigroups of analytic functions.

This construction provides a set of important non-trivial examples for Operator Theory.
Semigroups of analytic functions.

Another mathematical construction closely related to Loewner theory is one-parametric semigroups of analytic functions.

**Definition**

A *(one-parametric) semigroup of analytic functions* in $\mathbb{D}$ is a family $(\phi_t)_{t \geq 0} \subset H(\mathbb{D}, \mathbb{D})$ such that

1. **S1** $\phi_0 = \text{id}_{\mathbb{D}}$;
2. **S2** $\phi_{t+s} = \phi_t \circ \phi_s$;
3. **S3** $\phi_t(z) \to z$ as $t \to +0$ for any $z \in \mathbb{D}$.
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Another mathematical construction closely related to Loewner theory is one-parametric semigroups of analytic functions.

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These semigroups are a classical subject of study, as flows of continuous dynamical systems. Semigroups also appear in connection with the theory of Galton-Watson processes (branching processes). Furthermore, they provide the theory of continuous semigroups of operators (namely, between spaces of analytic functions) with an important factory of examples.
Semigroups of analytic functions.

**Theorem**

For any such semigroup \((\phi_t)\) there exists a holomorphic function \(G: \mathbb{D} \rightarrow \mathbb{C}\) such that

\[
G(z) = \lim_{t \to +0} \frac{\phi_t(z) - z}{t}, \quad \frac{d\phi_t(z)}{dt} = G(\phi_t(z)).
\]

\(G\) is known as the *infinitesimal generator* of the semigroup.
Semigroups of analytic functions.

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Theorem (E. Berkson, H. Porta, 1978)

\(G\) is an infinitesimal generator iff

\[
G(z) = (\tau - z)(1 - \overline{\tau}z)p(z),
\]

where \(\tau \in \overline{\mathbb{D}}\) and \(p : \mathbb{D} \to \mathbb{C}\) holomorphic and \(\text{Re } p \geq 0\).
Semigroups of analytic functions.

Theorem (E. Berkson, H. Porta, 1978)

$G$ is an infinitesimal generator iff $G(z) = (\tau - z)(1 - \overline{\tau}z)p(z)$, where $\tau \in \overline{D}$ and $p : D \to \mathbb{C}$ holomorphic and $\text{Re} \, p \geq 0$.

- If $\tau = 0$, then $G(z) = -zp(z)$, and the equation for $w(t) = \phi_t(z)$ is
  
  $$dw/dt = -wp(w),$$

  which is the autonomous version of the radial Loewner ODE;
Theorem (E. Berkson, H. Porta, 1978)

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- If $\tau = 1$, then $G(z) = (1 - z)^2p(z)$, and
  \[
  \frac{dw}{dt} = (1 - w)^2p(w),
  \]
  which resembles the autonomous chordal Loewner ODE.
Unified approach.

Our first contribution (with Bracci, Díaz-Madrigal, and Gumenyuk) to Loewner theory was to find a unified approach to the three settings we have mentioned. Our motivation: find a general construction which contains, as particular cases:

- radial Loewner evolution,
- chordal Loewner evolution,
- semigroups of analytic functions.

Three basic notions must be considered

- Evolution families \((\varphi_{s,t})\).
- Herglotz vector fields \(G(w, t)\).
- Loewner chains \((f_t)\).

In the literature there is not an intrinsic definition of EF or “chordal Loewner chain” (as in the radial case Pommerenke did).
What should an evolution family be?

There have been several attempts to give a definition of EF: 
Ω a simply connected domain (\( \mathbb{D} \), \( \mathbb{H} \), ...)
\( \phi_{s,t} : \Omega \to \Omega \), holomorphic (in fact, univalent), \( 0 \leq s \leq t < +\infty \):

**EF1**  \( \phi_{s,s} = \text{id}_\Omega \),

**EF2**  \( \phi_{s,t} = \phi_{u,t} \circ \phi_{s,u} \), whenever \( 0 \leq s \leq u \leq t < +\infty \),

**EF3**  ?
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\textbf{EF3} ?

\textbf{EF3: Radial case} 
\(\varphi_{s,t}(0) = 0\) and 
\(\varphi'_{s,t}(0) = e^{s-t}\).

\textbf{EF3: Chordal case} 
\(\varphi_{s,t}(z) = z - \frac{t-s}{z} + o(1/z)\)  
(hydrodynamic normalization at \(\infty\)).

\textbf{EF3: Semigroup case} 
\(\phi_t(z) \rightarrow z\) as \(t \rightarrow +0\) for any \(z \in \mathbb{D}\).

In this case, 
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**EF3:** Semigroup case
\(\phi_t(z) \to z\) as \(t \to +0\)
for any \(z \in \mathbb{D}\).
In this case,
\(\varphi_{s,t} := \phi_{t-s}\).

**Problem**

Assume \((\varphi_{s,t})\) satisfies EF1, EF2, and EF3.
Does \((\varphi_{s,t})\) come from a Loewner ODE?
What should an evolution family be?

Definition

A family \((\varphi_{s,t}), \ 0 \leq s \leq t < +\infty,\) of holomorphic self-maps of \(\mathbb{D}\) is a (generalized) evolution family of order \(d \in [1, +\infty]\) if

- **EF1** \(\varphi_{s,s} = \text{id}_{\mathbb{D}},\)
- **EF2** \(\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}\) whenever \(0 \leq s \leq u \leq t < +\infty,\)
- **EF3** for any \(z \in \mathbb{D}\) and \(T > 0\) there is \(k_{z,T} \in L^d([0, T], \mathbb{R})\) s. t.

\[
|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi) d\xi, \quad 0 \leq s \leq u \leq t \leq T.
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\]

When \(d = \infty\)

**EF3** for any \(z \in \mathbb{D}\) and \(T > 0\) there is a constant \(c_{z,T}\) s. t.

\[
|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq c_{z,T}(t - u), \ 0 \leq s \leq u \leq t \leq T.
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\[
|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi) d\xi, \quad 0 \leq s \leq u \leq t \leq T.
\]

In all the cases mentioned before, \((\varphi_{s,t})\) satisfies this definition with \(d = \infty\).
What should an evolution family be?

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A family \((\varphi_{s,t}), 0 \leq s \leq t < +\infty\), of holomorphic self-maps of \(\mathbb{D}\) is a (generalized) evolution family of order \(d \in [1, +\infty]\) if

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EF3 for any \(z \in \mathbb{D}\) and \(T > 0\) there is \(k_{z,T} \in L^d([0, T], \mathbb{R})\) s. t.

\[
|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_{u}^{t} k_{z,T}(\xi) d\xi, \quad 0 \leq s \leq u \leq t \leq T.
\]

Theorem

\(\varphi_{s,t}\) is univalent.
A (generalized) Herglotz function of order $d \in [1, +\infty]$ is a function $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ satisfying

**HF1** For all $z \in \mathbb{D}$, the function $t \in [0, +\infty) \mapsto p(z, t) \in \mathbb{C}$ belongs to $L^d_{loc}([0, +\infty), \mathbb{C})$;

**HF2** $p(\cdot, t)$ is holomorphic in $\mathbb{D}$ for a.e. $t \geq 0$;

**HF3** For all $z$ and for all $t$, we have $\text{Re } p(z, t) \geq 0$. 


Main result

**Theorem**

There is an essentially 1-to-1 correspondence between evolution families $(\varphi_{s,t})$ of order $d$ and couples $(\tau, p)$ where $
abla \tau : [0, +\infty) \rightarrow \mathbb{D}$ is a measurable function and $p$ is a Herglotz function of order $d$. 

---

Definition $G(w, t) := (w - \tau(t))(\tau(t)w - 1)p(w, t)$ is a (generalized) Herglotz vector field.
Main result

Theorem

There is an essentially 1-to-1 correspondence between evolution families \((\varphi_{s,t})\) of order \(d\) and couples \((\tau, p)\) where \(\tau : [0, +\infty) \rightarrow \overline{\mathbb{D}}\) is a measurable function and \(p\) is a Herglotz function of order \(d\).

That is, given \((\varphi_{s,t})\) there is \((\tau, p)\) such that for all \(s \geq 0\) and \(z \in \mathbb{D}\), the map \(t \in [s, +\infty) \mapsto w(t) := \varphi_{s,t}(z)\) is the solution for the Loewner ODE

\[
\frac{dw}{dt} = (w - \tau(t))(\overline{\tau(t)}w - 1)p(w, t), \quad w(s) = z. \tag{5.1}
\]

Conversely, given \((\tau, p)\), solving (5.1), \(\varphi_{s,t}(z) = w(t)\) is an EF.

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This is the main result of
(Generalized) Loewner chains

**Definition**

A family \((f_t)_{0 \leq t < +\infty}\) of holomorphic maps of the unit disc is called a \((generalized)\) Loewner chain of order \(d \in [1, +\infty]\) if

- **LC1** each function \(f_t : \mathbb{D} \rightarrow \mathbb{C}\) is univalent,
- **LC2** \(f_s(\mathbb{D}) \subset f_t(\mathbb{D})\) whenever \(0 \leq s < t < +\infty\),
- **LC3** for any compact set \(K \subset \mathbb{D}\) and all \(T > 0\) there exists \(k_{K,T} \in L^d([0, T], \mathbb{R})\) such that

\[
\sup_{z \in K} |f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\xi) d\xi, \quad 0 \leq s \leq t \leq T.
\]
(Generalized) Loewner chains

**Theorem**

- For any Loewner chain \((f_t)\) of order \(d\), the formula
  \[
  \varphi_{s,t} = f_t^{-1} \circ f_s
  \]
  defines an evolution family of the same order \(d\).

- For any evolution family \((\varphi_{s,t})\) of order \(d\), there exists a unique Loewner chain \((f_t)\) of the same order \(d\) s.t.:
  \(a\) \(\varphi_{s,t} = f_t^{-1} \circ f_s\) for any \(t \geq s \geq 0\);
  \(b\) \(f_0 \in S\), and
  \(c\) \(\Omega := \bigcup_{t \geq 0} f_t(\mathbb{D}) = \{z : |z| < R\},\ R \in (0, +\infty]\).
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Any other Loewner chain satisfying (a) is of the form \((g_t) = (F \circ f_t)\), where \(F : \Omega \to \mathbb{C}\) is univalent.
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Scheme of correspondence

Loewner chains \((f_t)\) \[ \varphi_{s,t} = f_t^{-1} \circ f_s \]

Evolution families \((\varphi_{s,t})\)

\[
\frac{\partial f_t(z)}{\partial t} = -G(z, t) \frac{\partial f_t(z)}{\partial z}
\]

Loewner ODE

\[
\frac{dw}{dt} = G(w, t), \quad w(s) = z
\]

\[
\varphi_{s,t}(z) = w(t)
\]

Herglotz vector fields \(G(w, t)\)

where \(G(w, t) = (w - \tau(t))(\overline{\tau(t)}w - 1)p(w, t)\).
Scheme of correspondence

Take $(\tau, p)$ and $G(w, t) = (w - \tau(t))(\tau(t)w - 1)p(w, t)$.

- If $G(z, t) = (\tau - z)(1 - \bar{\tau}z)p(z, t)$, where $\tau = 0$ and $p(0, t) = 1$ for a.e. $t \geq 0$, then we obtain the classical radial case.
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- If $G(z, t) = (\tau - z)(1 - \bar{\tau}z)p(z, t)$, where $\tau = 1$ and $p(z, t)$ satisfies some regularity condition at $z = 1$, then we obtain the classical chordal case.
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- If $G(z, t) = G(z)$ does not depend on $t$, then the evolution family we obtain is $\varphi_{s,t} = \phi_{t-s}$, where $(\phi_t)$ is the semigroup generated by $G$. 
Now, I present a list of open problems I am interested in.
Problem: New univalence criteria

As we have studied in the first part of this course, Loewner PDE can have more than one family of solutions and many of them are formed by non-univalent functions.
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Problem

*Find criteria to ensure that a solution of the general (or chordal) Loewner PDE is univalent.*

*Obtain new univalence criteria and characterization of starlike functions from a boundary point.*
Problem: Extensions of Loewner chain beyond $D$

An interesting problem is to analyze when the solution of Loewner’s equations admit quasiconformal extensions beyond the unit disk. This is a fundamental tool to apply Loewner theory to fluid dynamics (see the book by Gustafsson and Vasiliev).
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In 1972, J. Becker proved that if $p$ is a Herglotz function and there are numbers $0 < a < b$ such that

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Problem

*Find sufficient and necessary conditions on a Herglotz vector field that ensure that the associated Loewner chain and/or the evolution family are “regular” beyond the unit disk.*
Problem: Chordal Loewner chain

A Loewner chain is chordal when its associated Herglotz vector field has a factorization of the type $G(w, t) = (w - 1)^2 p(w, t)$. In the paper -M. D. C., S. Díaz-Madrigal, and P. Gumenyuk, *Geometry behind chordal Loewner chains*, Complex Anal. Oper. Theory 4 (2010), 541–587, we give some geometrical conditions to ensure a certain Loewner chain is chordal, but we do not have a characterization.
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we give some geometrical conditions to ensure a certain Loewner chain is chordal, but we do not have a characterization.

**Problem**

*Give an intrinsic characterization of chordal Loewner chains.*

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This problem for evolution families was solved (all the functions $\varphi_{s,t}$ must share the Denjoy-Wolff point and it must be at the boundary of $\mathbb{D}$) in
Problem: Stochastic approach

Without doubt, the success of the chordal equation is due to the many applications of its stochastic version. As we have seen, the starting point of Schramm development was to replace the driving term $h$ by a Brownian motion in

$$\frac{\partial k_t(w)}{\partial t} = -k'_t(w) \frac{2}{w - h(t)}, \quad (w \in \mathbb{H}).$$

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Problem

**Analyze the evolution family and the Loewner chain with a vector field**

$$G(z, t) = (z - \tau(t)) \overline{\tau(t)}(z - 1)p(z, t),$$

where $\tau : [0, \infty) \to \overline{\mathbb{D}}$ is a Brownian motion. One can begin by the case $p \equiv 1$. 
Problem: Loewner Theory in multiply connected domains

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Direct applications of the ideas from Loewner theory to families of univalent functions in multiply connected domains leads to a very trivial construction. Let us analyze a particular case: Take $0 < r < 1$ and $D = \{ z : r < |z| < 1 \}$. Then the connected component of $H(D, D)$ containing $\text{id}_D$ is the family of all rotations.
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Then

$$\varphi_{s,t}(z) = e^{(\lambda(s) - \lambda(t))i}z \quad \text{and} \quad f_s(D) = f_t(D).$$

Everything would be trivial!
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How can we study an evolution of domains like the following?
Problem: Loewner Theory in multiply connected domains

In the double connected case, this problem has been deal with by

**Slit case:** Komatu (1943, 1950), Goluzin (1951), Li En Pir (1953), Lebedev (1955), Kuvaev and Kufarev (1955).

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**Problem**

*Extend Loewner Theory to multiply connected domains with* $n > 2$. 