Extensions of T_0-quasi-metrics

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Preliminaries

Let X be a set and let $d : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then d is called a *quasi-pseudometric* on X if
(a) $d(x, x) = 0$ whenever $x \in X$, and
(b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

For any $x, y \in X$ we set $x \leq_d y$ if $d(x, y) = 0$ and call $\leq_d$ the *specialization (pre)order* of d.

Furthermore we shall say that a quasi-pseudometric d is a *T_0-quasi-metric* provided that d also satisfies the following *T_0-condition*: For each $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$.

Preliminaries Continued

For a quasi-pseudometric d on a set X obviously we have that $\leq_d$ is a partial order if and only if d is a T_0-quasi-metric.

Let d be a quasi-pseudometric on a set X, then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric, called the *conjugate* or *dual quasi-pseudometric of* d.

Moreover a quasi-pseudometric d on X such that $d = d^{-1}$ is called a *pseudometric*. Observe that for any quasi-pseudometric (resp. T_0-quasi-metric) d, $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ is a pseudometric (resp. metric).

Preliminaries Continued

For each $x, y \in \mathbb{R}$ we put $x \dot{-} y = \max\{x - y, 0\}$. Setting $u(x, y) = x \dot{-} y$ whenever $x, y \in \mathbb{R}$ we obtain a natural example of a T_0-quasi-metric space $(\mathbb{R}, u)$. Note that $\leq_u$ is the usual order on $\mathbb{R}$ and u^s is the usual metric on $\mathbb{R}$.

Let m be a metric and let $\leq$ be a partial order on a set X. Define for any $x, y \in X$, $r(x, y) = 0$ if $x \leq y$ and $r(x, y) = m(x, y)$ otherwise. Also set for any $x, y \in X$,
$D_{(m,\leq)}(x, y) = \inf\{\sum_{i=0}^{n-1} r(x_i, x_{i+1}) : n \in \omega, x_0, \ldots, x_{n-1}, x_n \in X, x_0 = x \text{ and } x_n = y\}$.

It is well known and obvious that $D_{(m,\leq)}$ is the largest quasi-pseudometric d on X such that $d \leq m$ and $\leq \subseteq \leq_d$.

Crucial Definitions

Let $(X, m, \leq)$ be a partially ordered metric space. Gaba and Künzi called a T_0-quasi-metric d on the set X *m-splitting* provided that $d \vee d^{-1} = m$. Furthermore they said that d is $(X, m, \leq)$ *producing* provided that d is m-splitting and the specialization order $\leq_d$ of d is equal to $\leq$.

Obviously each T_0-quasi-metric d on X produces the partially ordered metric space $(X, d^s, \leq_d)$.

Basic Results

[GabaKunzi] Given a partially ordered metric space $(X, m, \leq)$, that space is produced by a T_0-quasi-metric on X if and only if $D_{(m,\leq)}$ produces $(X, m, \leq)$; moreover $D_{(m,\leq)}$ is then the largest T_0-quasi-metric on X producing $(X, m, \leq)$.

Uniquely producing T_0-quasi-metrics

[Gaba/Künzi] A totally ordered metric space $(X, m, \leq)$ is produced by a T_0-quasi-metric if and only if it satisfies the interval condition, that is for any $x, y, z \in X$ with $x \leq y \leq z$ we have that $\max\{m(x, y), m(y, z)\} \leq m(x, z)$.

Furthermore such a totally ordered metric space $(X, m, \leq)$ is then uniquely produced by a T_0-quasi-metric, since evidently the only producing T_0-quasi-metric d satisfies $d(x, y) = 0$ if $x \leq y$ and $d(x, y) = m(x, y)$ otherwise.

A Counterexample

Example Take $X = \{0,1\} \times \{0,1\}$ be equipped with its maximum metric

$$m((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

and with its product order $\preceq$ defined by

$$(x_1, x_2) \preceq (y_1, y_2) \text{ provided that } x_1 \leq y_1 \text{ and } x_2 \leq y_2,$$

whenever $(x_1, x_2), (y_1, y_2) \in X$.

Counterexample Continued

It is possible to describe all T_0-quasi-metrics d on X that produce $(X, m, \preceq)$: Except for the two-element set $\{(0,1),(1,0)\}$ all other two-element sets in X are $\preceq$-chains. But any producing T_0-quasi-metric d on X is uniquely determined on a subchain, as noted above.

Furthermore by the m-splitting condition, one of the d-distances between $(0,1)$ and $(1,0)$ need to be 1, while for the other one we can choose any distance value ϵ in $(0,1]$. It remains to be checked that any d so defined indeed is a T_0-quasi-metric on X producing $(X, m, \preceq)$.

Uniquely Produced Space

Example Let v be the T_0-quasi-metric on the four element set $X = \{a, b, c, d\}$ (enumerated as listed) defined for any $x, y \in X$ by $v(x, y) = v_{xy}$ with the help of the matrix V:

$$\mathbf{V} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 \end{pmatrix}.$$

Example Continued

Then v is indeed a T_0-quasi-metric on X.
Note that the matrix M corresponding to the metric $m = v^s$ is as follows:

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 3 \\ 2 & 1 & 0 & 2 \\ 2 & 3 & 2 & 0 \end{pmatrix}.$$

Argument Continued

Assume now that q is a T_0-quasi-metric on X that produces the partially ordered metric space $(X, m, \leq_v)$:

Then the functions q and v have the same zeros, because $\leq_v = \leq_q$. For all the other pairs of points distinct from (a, b) or (b, a) in $X \times X$ the value of q and v must agree, since $q^s = v^s$ and the conjugate of each such pair belongs to the partial order $\leq_v$ and thus is mapped to 0 by q.

Argument Continued

It remains to be seen that $q(a,b) = q(b,a) = 1$. But indeed this follows from the triangle inequality and the fact that $q^s(a,b) = 1$, because we have $3 = q(d,b) \leq q(d,a) + q(a,b) = 2 + q(a,b)$ and $2 = q(c,a) \leq q(c,b) + q(b,a) = 1 + q(b,a)$. Hence $q(a,b) = q(b,a) = 1$ and so necessarily $q = v$.

Therefore v is indeed the unique T_0-quasi-metric on X that produces the partially ordered metric space $(X, m, \leq_v)$. However the specialization order $\leq_v$ of v is not total, since a and b are not $\leq_v$-comparable.

Some Further Definitions*

Let $(X, m, \leq)$ be a partially ordered metric space.
A T_0-quasi-metric d on X with $d^s = m$ is called *minimally m-splitting* if whenever q is a quasi-pseudometric on X that is m-splitting and $q \leq d$, then $q = d$.

A T_0-quasi-metric d on a partially ordered metric space $(X, m, \leq)$ is called *minimally producing* if d produces $(X, m, \leq)$ and whenever q is a quasi-pseudometric on X such that $q \leq d$ and q produces $(X, m, \leq)$, then $q = d$.

A partial order $\leq$ on X is called *maximally m-produced* if whenever $\preceq$ is a partial order on X such that $\leq \subseteq \preceq$ and both $(X, m, \leq)$ and $(X, m, \preceq)$ are produced by T_0-quasi-metrics on X, then $\leq = \preceq$.

Some Further Observations*

Lemma

Let $(X, m, \leq)$ be a partially ordered metric space. Then $D_{(m,\leq)}$ is minimally producing $(X, m, \leq)$ if and only if $D_{(m,\leq)}$ is the unique producing T_0-quasi-metric on $(X, m, \leq)$.

[Gaba/Künzi] Let (X, m) be a metric space. Then a T_0-quasi-metric d on X is minimally m-splitting if and only if d is minimally producing $(X, m, \leq_d)$.

Further Observations Continued*

Proposition

Let $(X, m, \leq)$ be a partially ordered metric space. Then the following conditions are equivalent:
(a) $D_{(m,\leq)}$ is the unique T_0-quasi-metric producing $(X, m, \leq)$.
(b) $D_{(m,\leq)}$ is minimally m-splitting and $\leq = \leq_{D_{(m,\leq)}}$.
(c) $D_{(m,\leq)}$ is minimally m-splitting and $\leq$ is maximally m-produced.

Some Final Observations*

Proposition

Let $(X, m, \leq)$ be a finite partially ordered metric space. Then $D_{(m,\leq)}$ is minimally m-splitting if and only if it is uniquely producing $(X, m, \leq)$.

Corollary

Let $(X, m, \leq)$ be a partially ordered metric space such that $D_{(m,\leq)}$ is minimally m-splitting. Then $(X, m, \leq_{D(m,\leq)})$ is uniquely produced by a T_0-quasi-metric on X.

Another Example*

Example We define a metric m on $X = \{1, 2, 3\}$ by the matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}, \text{that is, } m(i,j) = m_{i,j}.$$

Example Continued*

Consider the partial order $\leq = \{(1,2),(1,3)\} \cup \{\Delta_X\}$ on X where Δ_X denotes the diagonal of X. Then $(X, m, \leq)$ is produced by a T_0-quasi-metric. In fact it is readily checked that the largest T_0-quasi-metric $D_{(m,\leq)}$ producing $(X, m, \leq)$ is given by the matrix

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}.$$

Example Continued*

One finds a minimally m-splitting T_0-quasi-metric $s < D_{(m,\leq)}$ on X given by the matrix

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 1 & 0 \end{pmatrix}.$$

The specialization orders of $D_{(m,\leq)}$ and s are equal to $\leq$. Hence s is minimally producing $(X, m, \leq)$, but s is not uniquely producing $(X, m, \leq)$.

Second Part of Talk

Problem Let $(X, m, \leq)$ be a partially ordered metric space and A a subset of X.

(1) Suppose that d is T_0-quasi-metric on A which is $m|(A \times A)$-splitting. When can d be extended to an m-splitting T_0-quasi-metric $\widetilde{d}$ on X?

(2) Suppose that d is a T_0-quasi-metric on A which is $(A, m|(A \times A), \leq |(A \times A))$-producing. When can d be extended to a T_0-quasi-metric $\widetilde{d}$ on X that produces $(X, m, \leq)$?

A Counterexample: Extending Splitting T_0-quasi-metrics

Example

Let $A = \{a, b, c, d\}$ be the 4-element chain $a < b < c < d$. Endow A with the T_0-quasi-metric q defined for $x_1, x_2 \in A$ by $q(x_1, x_2) = 0$ if $x_1 \leq x_2$ and $q(x_1, x_2) = 5$ otherwise. Set $m(x_1, x_2) = q^s(x_1, x_2)$ whenever $x_1, x_2 \in A$.
For some point $e \notin A$ set $X = A \cup \{e\}$. We define $m(d, e) = m(e, d) = 6$, $m(e, c) = m(c, e) = 7$, $m(b, e) = m(e, b) = 3$, and $m(e, a) = m(a, e) = 4$, and $m(e, e) = 0$.
Then (X, m) is a metric space.
One can show that there does not exist an m-splitting T_0-quasi-metric extension $\widetilde{q}$ of q from A to X.

A Crucial Result of Deák

Deák proved the following result:
Let X be a set and A a (nonempty) subset of X, d a quasi-pseudometric on A, and e a quasi-pseudometric on X. For any $x, y \in X$ set

$$d_e(x, y) = \min\{e(x, y), \inf_{a,b \in A}\{e(x, a) + d(a, b) + e(b, y)\}\}.$$

Finally assume that $d \leq e|(A \times A)$.

Then d_e is a quasi-pseudometric on X such that $d_e|(A \times A) = d$ and $d_e \leq e$.

Some First Observation*

Lemma

Let (X, e) be a quasi-pseudometric space and A be a nonempty subspace of X. Let d be a quasi-pseudometric on X such that $d \leq e$. Then $d \leq (d|(A \times A))_e$.

First Positive Results

Proposition

Let A be a subspace of a metric space (X, m) and suppose that there exists an $m|(A \times A)$-splitting T_0-quasi-metric d on A. Then we can extend d to the quasi-pseudometric d_m on X such that $d_m \leq m$.

If d' is another $m|(A \times A)$-splitting T_0-quasi-metric on A such that $d \leq d'$, then $d_m \leq (d')_m$.

A First Characterization

Proposition

Let A be a subspace of a metric space (X, m) and suppose that there exists an $m|(A \times A)$-splitting T_0-quasi-metric d on A. Furthermore suppose that d_m is defined as above. Then d_m is m-splitting if and only if for each $x, y \in X$ we have that

$$m(x, y) \leq \inf_{a,b \in A} \{m(x, a) + d(a, b) + m(b, y)\}$$

or

$$m(x, y) \leq \inf_{a,b \in A} \{m(y, a) + d(a, b) + m(b, x)\}.$$

One-point-extensions

Corollary

Let A be a subspace of the metric space (X, m) and suppose that d is a T_0-quasi-metric on A that is $m|(A \times A)$-splitting. Furthermore suppose that $X = A \cup \{\infty\}$ with $\infty \notin A$. Then d_m is m-splitting on X if and only if for each $x \in A$, we have that

$$m(x, \infty) \leq \inf_{b \in A} \{d(x, b) + m(b, \infty)\}$$

or

$$m(x, \infty) \leq \inf_{a \in A} \{m(\infty, a) + d(a, x)\}.$$

The Importance of d_m

Remark

Let A be a subspace of the metric space (X, m) and suppose that d is a T_0-quasi-metric on A that is $m|(A \times A)$-splitting. Suppose furthermore that there exists an m-splitting T_0-quasi-metric $\widetilde{d}$ on X that extends d from A to X.

Then $\widetilde{d} \leq d_m$.

Therefore d_m is an explicit description of the largest m-splitting T_0-quasi-metric extension of d from A to X (provided that such extensions exist).

An Illustration*

Example

Let $(X, m, \leq)$ be a partially ordered metric space produced by a T_0-quasi-metric d on X and let A be a closed subspace of (X, m). Then $\leq_{(d|(A\times A))_m} = \Delta_X \cup \leq_{d|(A\times A)}$.

The Final Characterization

Corollary

Let A be a subspace of a metric space (X, m) and suppose that there exists an $m|(A \times A)$-splitting T_0-quasi-metric d on A. Then there exists an m-splitting T_0-quasi-metric extension of d from A to X if and only if for each $x, y \in X$,

$$m(x, y) \leq \inf_{a,b \in A} \{m(x, a) + d(a, b) + m(b, y)\}$$

or

$$m(x, y) \leq \inf_{a,b \in A} \{m(y, a) + d(a, b) + m(b, x)\}.$$

Comparable Splitting T_0-quasi-metrics*

Corollary

Let A be a subspace of a metric space (X, m) and suppose that there exist $m|(A \times A)$-splitting T_0-quasi-metrics d_1 and d_2 with $d_1 \leq d_2$ on A. Then if d_1 has an m-splitting T_0-quasi-metric extension $\widetilde{d_1}$ from A to X, then d_2 has an m-splitting T_0-quasi-metric extension $\widetilde{d_2}$ from A to X.

Dense Subspaces

Proposition

Let A be a dense subset of a metric space (X, m) and let d be a T_0-quasi-metric on A that splits $m|(A \times A)$.

(a) Then d can be extended to an m-splitting T_0-quasi-metric $\widetilde{d}$ on X and there is no other m-splitting T_0-quasi-metric extension of d from A to X.

(b) Of course $\widetilde{d} = d_m$.

Example

In case that A is no longer dense in (X, m), uniqueness of an m-splitting T_0-quasi-metric extension of d from A to X is no longer guaranteed.

Example Let A be a set and $X = A \cup \{\infty\}$ with $\infty \notin A$. Moreover let (X, m) be the discrete metric space defined for $x, y \in X$ by $m(x, y) = 1$ if $x \neq y$, and $m(x, y) = 0$ otherwise.
Let B be any subset of A. Set $\widetilde{d}_B(x, y) = m(x, y)$ if $x, y \in A$. Furthermore set $\widetilde{d}_B(x, \infty) = 0$ if $x \in B \cup \{\infty\}$, $\widetilde{d}_B(x, \infty) = 1$ if $x \in X \setminus (B \cup \{\infty\})$ and $\widetilde{d}_B(\infty, x) = 1$ if $x \in A$.

One checks that $\widetilde{d}_B$ is a T_0-quasi-metric on X.
Moreover $\preceq_{\widetilde{d}_B} = \Delta_X \cup [B \times \{\infty\}]$ is obviously the specialization order of $\widetilde{d}_B$ on X.

Example Continued*

The partially ordered metric space $(A, m|(A \times A), \preceq_B |(A \times A))$ is produced by $m|(A \times A)$ and the partially ordered metric space $(X, m, \preceq_B)$ is produced by the T_0-quasi-metric $\widetilde{d_B}$ on X, which extends $m|(A \times A)$ from A to X.

Hence the $m|(A \times A)$-splitting T_0-quasi-metric $m|(A \times A)$ extends to each member of the family $(\widetilde{d_B})_{B \subseteq A}$ of m-splitting T_0-quasi-metrics on X.

Furthermore $m = \widetilde{d_\emptyset}$ is the largest such m-splitting extension of $m|(A \times A)$ from A to X.

New Problem: Extending Producing T_0-quasi-metrics

Suppose that A is a subspace of a partially ordered metric space $(X, m, \leq)$ and d is a T_0-quasi-metric on A producing $(A, m|(A \times A), \leq |(A \times A))$. A necessary condition in order that we can extend d to a producing T_0-quasi-metric on $(X, m, \leq)$ is obviously that $(X, m, \leq)$ possesses a producing T_0-quasi-metric (that is, equivalently, that $D_{(m,\leq)}$ produces $(X, m, \leq)$).

A Counterexample

Example Let $X = \{a, b\}$ be a two-element set with metric m defined by $m(b, a) = m(a, b) = 2$ and equal to 0 otherwise. Then obviously m produces the partially ordered metric space (X, m, Δ_X).

Now add a point $c \notin X$ with $a \leq c$ and $b \leq c$ and extend the metric m to $\{a, b, c\}$ by setting $m(x, c) = m(c, x) = 1$ whenever $x \in \{a, b\}$. Then the partially ordered metric space $(\{a, b, c\}, m, \leq)$ is not produced by a T_0-quasi-metric.
In particular one can show that $m|(X \times X)$ cannot be extended to a producing T_0-quasi-metric on $(\{a, b, c\}, m, \leq)$.

A Necessary Condition

Proposition

Let A be a subspace of a partially ordered metric space $(X, m, \leq)$ and d a T_0-quasi-metric on A producing $(A, m|(A \times A), \leq |(A \times A))$.
A necessary condition so that d can be extended to a producing T_0-quasi-metric on $(X, m, \leq)$ is that $d \leq D_{(m,\leq)}|(A \times A)$.

Another Necessary Condition

Suppose that $(X, m, \leq)$ is a partially ordered metric space and let A be a subset of X such that there exists a T_0-quasi-metric d on A which produces $(A, m|(A \times A), \leq |(A \times A))$.
Moreover suppose that there exists a T_0-quasi-metric $\widetilde{d}$ extending d from A to X which produces $(X, m, \leq)$.

Then $\widetilde{d}$ is an m-splitting T_0-quasi-metric extension of d from A to X, that is, d satisfies the corresponding conditions stated earlier.

Plan For This Section

Let a partially ordered metric space $(X, m, \leq)$ be given. Suppose that the T_0-quasi-metric d on A produces $(A, m|(A \times A), \leq |(A \times A))$.

We shall apply Deák's construction to d on A and the quasi-pseudometric $e = D_{(m,\leq)}$ on X.

Then $d_{D_{(m,\leq)}}$ will be a well-defined quasi-pseudometric on X, since Deák's hypothesis $d \leq D_{(m,\leq)}|(A \times A)$ will be satisfied if we assume that $(X, m, \leq)$ is produced by a T_0-quasi-metric on X.

A First Result

Proposition

Let $(X, m, \leq)$ be a partially ordered metric space and A a subset of X. Furthermore let d be a T_0-quasi-metric on A that produces $(A, m|(A \times A), \leq |(A \times A))$.

Then there exists a T_0-quasi-metric extension $\tilde{d}$ of d from A to X such that $\tilde{d}$ produces $(X, m, \leq)$ if and only if $d_{D_{(m,\leq)}}$ is the largest T_0-quasi-metric extension of d from A to X that produces $(X, m, \leq)$.

Characterization

Let $(X, m, \leq)$ be a partially ordered metric space, A a subset of X and d a T_0-quasi-metric on A that produces $(A, m|(A \times A), \leq |(A \times A))$. Then there exists a T_0-quasi-metric extension of d from A to X that produces $(X, m, \leq)$ if and only if the following two conditions are satisfied:
(1) for any $x, y \in X$ we have that

$$\min\{D_{(m,\leq)}(x,y), \inf_{a,b\in A}\{D_{(m,\leq)}(x,a)+d(a,b)+D_{(m,\leq)}(b,y)\}\} \geq m(x,y)$$

or

$$\min\{D_{(m,\leq)}(y,x), \inf_{a,b\in A}\{D_{(m,\leq)}(y,a)+d(a,b)+D_{(m,\leq)}(b,x)\}\} \geq m(x,y),$$

(2) if $x, y \in X$ and

$$\min\{D_{(m,\leq)}(x,y), \inf_{a,b\in A}\{D_{(m,\leq)}(x,a) + d(a,b) + D_{(m,\leq)}(b,y)\}\} = 0,$$

then $x \leq y$.

One-point-extensions

Let $(X, m, \leq)$ be a partially ordered metric space and $X = A \cup \{\infty\}$ where $\infty \notin A$. Furthermore let d be a T_0-quasi-metric on A that produces $(A, m|(A \times A), \leq |(A \times A))$. Then there exists a T_0-quasi-metric extension of d from A to X that produces $(X, m, \leq)$ if and only if the following two conditions are satisfied:

(1) for any $x \in A$ we have that

$$\inf_{b \in A}\{d(x, b) + D_{(m,\leq)}(b, \infty)\} \geq m(x, \infty)$$

or

$$\inf_{a \in A}\{D_{(m,\leq)}(\infty, a) + d(a, x)\} \geq m(x, \infty),$$

(2) if $x \in A$ and $\inf_{b \in A}\{d(x, b) + D_{(m,\leq)}(b, \infty)\} = 0$, then $x \leq \infty$; furthermore if $x \in A$ and $\inf_{a \in A}\{D_{(m,\leq)}(\infty, a) + d(a, x)\} = 0$, then $\infty \leq x$.

Comparable Producing T_0-quasi-metrics*

Corollary

Let $(X, m, \leq)$ be a partially ordered metric space and A be a subset of X. Furthermore let d_1 and d_2 be T_0-quasi-metrics on A satisfying $d_1 \leq d_2$ and producing $(A, m|(A \times A), \leq |(A \times A))$.

If there exists a T_0-quasi-metric extension $\widetilde{d_1}$ of d_1 from A to X such that $\widetilde{d_1}$ produces $(X, m, \leq)$, then there exists a T_0-quasi-metric extension $\widetilde{d_2}$ of d_2 from A to X such that $\widetilde{d_2}$ produces $(X, m, \leq)$.

Another Observation

Remark

Let $(X, m, \leq)$ *be a partially ordered metric space and let* A *be a dense subspace of* (X, m). *Furthermore let* d *be a* T_0*-quasi-metric on* A *such that* d *produces* $(A, m|(A \times A), \leq|(A \times A))$.

If $\widetilde{d}$ *denotes the unique* m*-splitting* T_0*-quasi-metric extension of* d *from* A *to* X, *then* $(X, m, \leq_{\widetilde{d}})$, *which is produced by* $\widetilde{d}$, *is obviously the unique partially ordered metric space on* X *which is produced by a* T_0*-quasi-metric on* X *that extends* d *from* A *to* X.

Yet Another Illustrating Example

Let $A = \{\frac{-1}{n}, \frac{1}{n} : n \in \mathbb{N}\}$, $X = A \cup \{0\}$, and $m = u^s|(X \times X)$. Choose $\preceq = \Delta_X \cup (\{0\} \times X)$ as the partial order on X. Note that A is a dense subset of (X, m).
Then $(X, m, \preceq)$ is not produced by a T_0-quasi-metric on X, although $(A, m|(A \times A), \preceq |(A \times A))$ is produced by the metric $m|(A \times A)$, since $\preceq |(A \times A)$ is equality.

Clearly m is the unique m-splitting T_0-quasi-metric extension $\widetilde{d}$ of $m|(A \times A)$ from A to X.
Of course, the specialization order of m is equality, that is, distinct from $\preceq$.

Final Example

Let A be a dense subset of a metric space (X, m).

Suppose that d_1 and d_2 are two T_0-quasi-metrics on A that both produce $(A, m|(A \times A), \Delta_A)$.

Our final example will show that the specialization orders of the m-splitting T_0-quasi-metric extensions $\widetilde{d_1}$ of d_1 and $\widetilde{d_2}$ of d_2 from A to X can be distinct.

Final Example Continued

Let $X = A \cup \{0\}$ where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, and $\preceq = \Delta_X \cup \{(0,1)\}$. Furthermore let $d(x,y) = x$ if $x \in X, y = 1$, and $x \neq y$. Finally set $d(x,y) = |x - y|$ otherwise.

One verifies that d is a T_0-quasi-metric on X. The specialization order of d is $\Delta_X \cup \{(0,1)\} = \preceq$. Furthermore d^s is the usual metric m on X.

Both the T_0-quasi-metrics $d|(A \times A)$ and $m|(A \times A)$ produce the partially ordered metric space $(A, m|(A \times A), \Delta_A)$. But the extension m on X of $m|(A \times A)$ produces (X, m, Δ_X), while the extension d on X of $d|(A \times A)$ produces $(X, m, \preceq)$.

Some References

References

[1] J. Deák, Extending a quasi-metric, Studia Sci. Math. Hungar. 28 (1993), no. 1-2, 105–113.

[2] P. Fletcher and W.F. Lindgren, Quasi-uniform Spaces, Dekker, New York, 1982.

[3] Y.O.K. Ulrich Gaba, Construction of quasi-metrics determined by orders, PhD thesis, University of Cape Town, South Africa, 2016.

Some References Continued II

[4] Y.U. Gaba and H.-P.A. Künzi, Splitting metrics by T_0-quasi-metrics, Topology Applications 193 (2015), 84–96.

[5] Y.U. Gaba and H.-P.A. Künzi, Partially ordered metric spaces produced by T_0-quasi-metrics, Topology Applications 202 (2016), 366–383.

[6] H.-P.A. Künzi, An introduction to quasi-uniform spaces, in: Beyond Topology, eds. F. Mynard and E. Pearl, Contemp. Math., Amer. Math. Soc. 486 (2009), pp. 239–304.

The End

THANK YOU!