Ultracomplete spaces.

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22 de junio de 2016. Valencia, España

Definition

A space X is called strongly complete if $\chi(X, \beta X) \leq \omega$.

Definition

(Romaguera) A space X is called cofinally Čech-complete if there is a countable collection $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of X satisfying the property that whenever $\mathcal{F}$ is filter on X such that for each $n \in \mathbb{N}$ there is some $G_n \in \mathcal{G}_n$ which meets all the members of $\mathcal{F}$, then $\mathcal{F}$ has a cluster point.

Theorem

([?], Theorem 2). *A metrizable space admits a cofinally complete metric if and only if it is cofinally Čech-complete.*

- Buhagiar and Yoshioka proved in 2001 that strongly completeness is equivalent to cofinal Čech-completeness, and renamed it ultracompleteness.

1. $nlc(X) = \{x \in X : X \text{ is not locally compact at } x\}$
2. **Definition** (Jardón, Tkachuk, 2003) A space X is said to be almost locally compact if there is a compact $K \subset X$ such that $\chi(K, X) \leq \omega$ and $nlc(X) \subset K$.
3. In this work we summarize properties of ultracomplete and almost locally compact spaces.
   (1987) Ponomarev, Tkachuk, *The countable character of X in βX compared with the countable character of the diagonal in $X \times X$*.
   (1999) A. García-Máynez, S. Romaguera, *Perfect pre-images of cofinally complete metric spaces*.
   (2001) D. Buhagiar, I. Yoshioka, *Ultracomplete topological spaces*.

Theorem

For space X, the following conditions are equivalent:
i) X is ultracomplete
ii) $\chi(X, cX) \leq \omega$ for some compactification cX of X;
iii) $\chi(X, kX) \leq \omega$ for every compactification kX of X;
iv) $cX \setminus X$ is hemicompact for some compactification cX of X;
v) $kX \setminus X$ is hemicompact for every compactification kX of X.

García Máynez-Romaguera and Ponomarev-Tkachuk proved the next:

Theorem

If X is an ultracomplete space, then $nlc(X)$ is bounded in X.

which implies that an ultracomplete space without points of local compactness is pseudocompact.

Theorem

(G-R and P-T) A paracompact space X is ultracompete if and only if it is almost locally compact.

Theorem

For any space X, the following conditions are equivalent:
i) X is almost locally compact;
ii) $cX \setminus X$ is locally compact and Lindelöf for some compactification cX of X;
iii) $kX \setminus X$ is locally compact and Lindelöf for every compactification kX of X.

From definitions we have the implications: locally compact $\Rightarrow$ almost locally compact $\Rightarrow$ ultracomplete $\Rightarrow$ Čech-complete

Example

i) The set of all irrationals numbers with its natural topology induced from $\mathbb{R}$ is a non-ultracomplete Čech-complete space.
ii) ω_1^ω is ultracomplete and it is not almost locally compact. $(nlc(\omega_1^\omega) = \omega_1^\omega)$
iii) The space $X = [0,1] \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$ is almost locally compact and it is not locally compact. $(nlc(X) = \{0\})$

Theorem

Let X be an infinite compact space. If $F(X)$ is the Markov free topological group of X, then $Y = \beta(F(X)) \setminus F(X)$ is an ultracomplete space without points of local compactness.

An open continuous image of a Čech-complete space is not necessarily Čech-complete. Any perfect image and any perfect preimage of a Čech-complete space is Čech-complete.

Theorem

i) The open continuous image of an ultracomplete (almost locally compact) space is ultracomplete (almost locally compact).
ii) Let f be a perfect function from a space X onto a space Y. Then X is ultracomplete (almost locally compact) if and only if Y is ultracomplete (almost locally compact).

Theorem

Let X and Y be two ultracomplete spaces. Then $X \times Y$ is ultracomplete if and only if one of the following conditions holds:
i) X and Y are locally compact,
ii) either X or Y is countably compact, locally compact,
iii) both X and Y are countably compact.

Theorem

Let X and Y be two almost locally compact spaces. Then $X \times Y$ is almost locally compact if and only if either
i) X and Y are locally compact, or
ii) either X or Y is compact.

Theorem

Let X_n be an ultracomplete, countably compact space for every $n \in \mathbb{N}$, then $X = \prod_{n\in\mathbb{N}} X_n$ is ultracomplete countably compact.

The space $X = ([0,1] \setminus \{\frac{1}{n} : n \in \mathbb{N}\}) \times \mathbb{N}$ is the union of two locally compact spaces, and it is not almost locally compact.

Theorem

i) If X is a metrizable space, $n \in \mathbb{N}$ and $X^n = X_1 \cup X_2 \cup \cdots \cup X_n$, where every X_k is ultracomplete then X is also ultracomplete.
ii) If X^ω is a countable union of ultracomplete subspaces, then the space X^ω is ultracomplete.
iii) For any space X and $n \in \mathbb{N}$, if $X^n = X_1 \cup \cdots \cup X_n$ and every X_k is almost locally compact then X is almost locally compact.

Problem

Suppose that $X^2 = X_1 \cup X_2$, where X_1 and X_2 are ultracomplete. Must X be ultracomplete?

If X is a topological space, $\mathcal{K}(X)$ ($\mathcal{K}_C(X)$) denotes the hyperspace of all nonempty compact (compact connected) subsets of X endowed with the Vietoris topology.

Example

If $X = [0,1] \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$, then:
i) X and $\mathcal{K}_C(X)$ are almost locally compact,
ii) $\mathcal{K}(X)$ and $\mathcal{F}_n(X)$ are not almost locally compact for all $n > 1$,

Theorem

i) The hyperspace $\mathcal{K}(X)$ is almost locally compact if and only if X is locally compact.
ii) If the hyperspace $\mathcal{K}(X)$ is ultracomplete, then X is ultracomplete and either countably compact or locally compact.

Example

i) In Theorem 3.15 of [**?**] it was stablished the existence a dense countable subspace of $\{0,1\}^{\mathfrak{c}}$ without nontrivial convergent sequences. If D is such subspace, then it was proved in Example 3.16 of [**?**] that $X = \{0,1\}^{\mathfrak{c}} \setminus D$ is ultracomplete non-countably compact and has no points of local compactness.
ii) $\mathcal{K}(X)$ is not ultracomplete.

Problem

i) Suppose that X is countably compact, ultracomplete with $nlc(X) = X$. Is it true that $\mathcal{K}(X)$ is ultracomplete?
ii) Is it true that $\mathcal{K}(\omega_1^{\omega})$ is ultracomplete?

Theorem

(Romaguera, Sanchis) If X is a topological group, then X is ultracomplete if and only if it is locally compact.

Theorem

A function space $C_p(X)$ has a dense ultracomplete subspace if and only if X is finite.

Theorem

Suppose that X is paracompact, (metalindelof, Dieudonné complete, splittable, Eberlein-Grothendieck, subspace of a Corson compact, subspace of $C_p(Y)$, where Y is Lindelöf Σ or pseudocompact. If X is ultracompete, then it is almost locally compact.

ω_1^ω is an ultracomplete homogeneous space and it is not almost locally compact.

Problem

Suppose that X is ultracomplete and the closure of every countably compact subspace of X is compact. Must X be almost locally compact?

Problem

i) Suppose that X is countably compact, ultracomplete with $nlc(X) = X$. Is it true that $\mathcal{K}(X)$ is ultracomplete?
ii) Is it true that $\mathcal{K}(\omega_1^\omega)$ is ultracomplete?

Problem

Suppose that $X^2 = X_1 \cup X_2$, where X_1 and X_2 are ultracomplete. Must X be ultracomplete?