Group actions on hyperspaces of convex sets and Banach-Mazur compacta

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Preliminary definitions

- Given a Banach space L and $K \subset L$ a convex subset, we denote by $cc(L)$ the hyperspace of all compact convex subsets of K equipped with the Hausdorff metric:

$$d_H(A, B) = \text{máx}\left\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\right\},$$

where $d(b, A) = \text{ínf}\{d(b, a) \mid a \in A\}$.

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where $d(b, A) = \text{ínf}\{d(b, a) \mid a \in A\}$.

- We denote by $\mathbb{B}^n$ the unitary euclidean ball, i.e.

$$\mathbb{B}^n = \{z \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

where $\|\cdot\|$ denotes the euclidean norm.

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- Let $cb(\mathbb{R}^n)$ be the subspace of $cc(\mathbb{R}^n)$ consisting of all convex bodies.

Banach Mazur Compacta

- The Banach-Mazur compactum $BM(n)$ is the set of isometry classes of n-dimensional Banach spaces topologized by the following metric best known in Functional Analysis as the Banach-Mazur distance:

$$d([E],[F]) = \ln \text{ínf}\{\|T\|\cdot\|T^{-1}\| \mid T : E \to F \text{ linear isomorphism}\}.$$

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- These spaces were introduced in 1932 by S. Banach and they continue to be of interest.
- For each $n \geq 2$, $BM(n)$ is an infinite dimensional compact AR.

Question (A. Pelczyński)

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- Open for $n \geq 3$.

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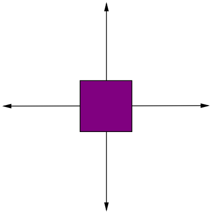

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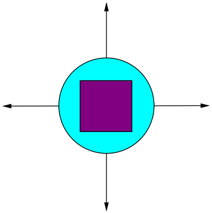

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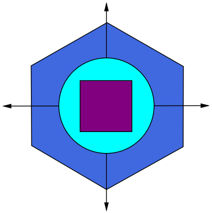

If $F = (\mathbb{R}^n, f)$ and $E = (\mathbb{R}^n, h)$ are n-dimensional Banach spaces, the isometry class of F, $[F]$ is the isometry class of E if and only if there is a linear transformation $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $g(B_f) = B_h$.

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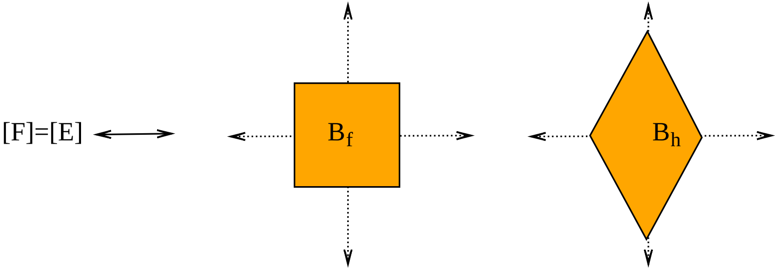

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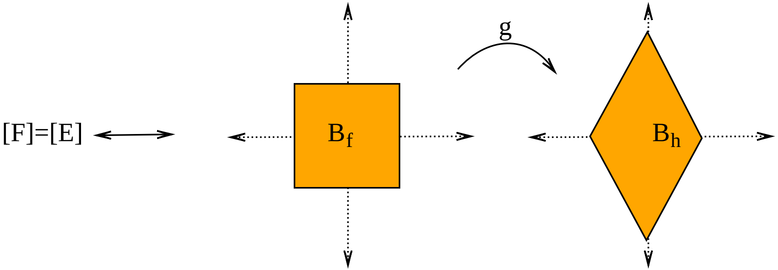

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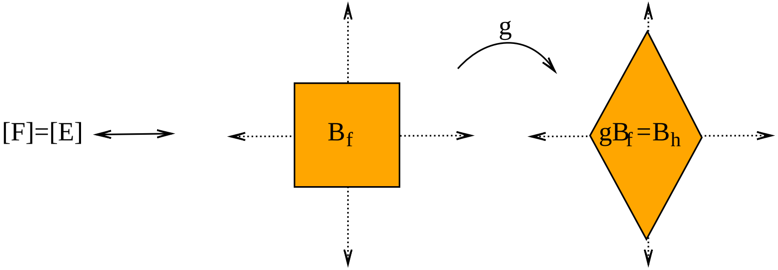

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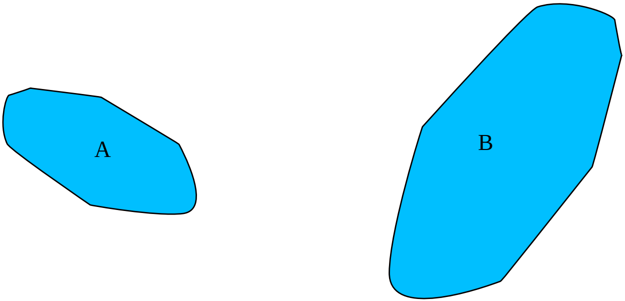
(S. Antonyan, 2000)

For every $n \geq 2$, $\mathcal{B}(n)$ is homeomorphic to $\mathbb{R}^p \times Q$, where Q denotes the Hilbert cube and $p = n(n+1)/2$.

In 1951 Macbeath studied the topology of the quotient space $cb(\mathbb{R}^n)/\sim$, where $A \sim B$ iff there exists an affine transformation g, such that $g(A) = B$.



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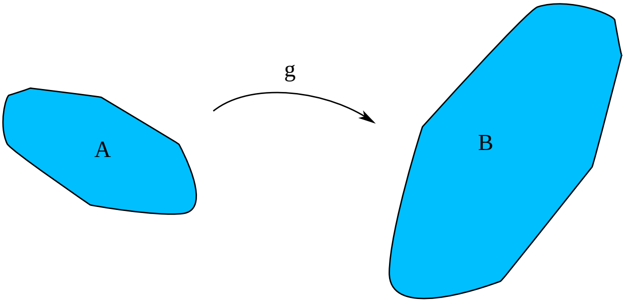

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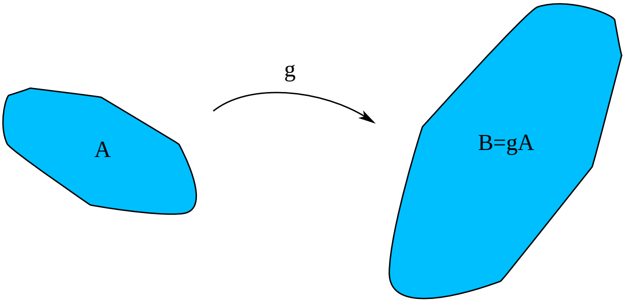

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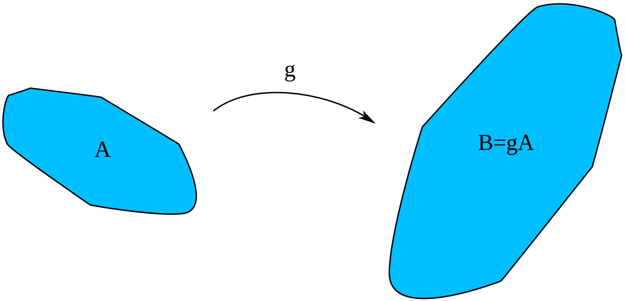

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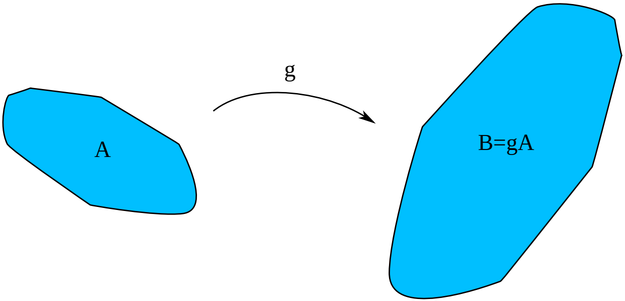

Macbeath proved that $cb(\mathbb{R}^n)/\sim$ is a compact metric space. However, for $n \geq 2$, the subspace $cb(\mathbb{R}^n)$ of $cc(\mathbb{R}^n)$ had not been computed.

Topological Transformation Groups

The equivalence relations $\sim$ defined above can be translated to the language of topological transformation groups.

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A continuous action of a topological group G on a topological space X is a map $\theta : G \times X \to X$ such that

1. $\theta(e, x) = x$,
2. $\theta(g, \theta(h, x)) = \theta(gh, x)$,

for all $x \in X$, $g, h \in G$.

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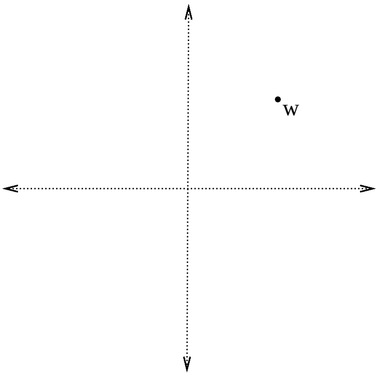
For every $g \in G$ and $x \in X$, the element $\theta(g, x)$ is denoted by gx.

Example

The group $\mathbb{S}^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}$ acts on $\mathbb{C}$ by the complex multiplication:

$$(z, w) \to zw$$



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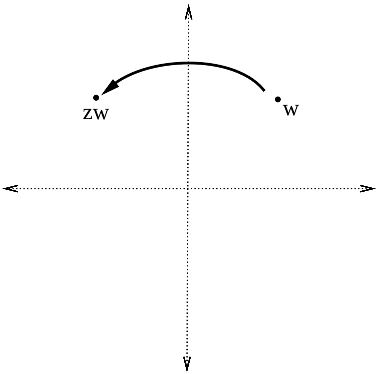

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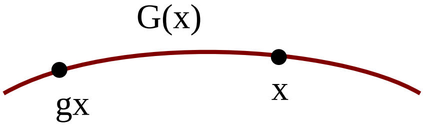

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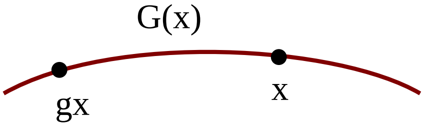
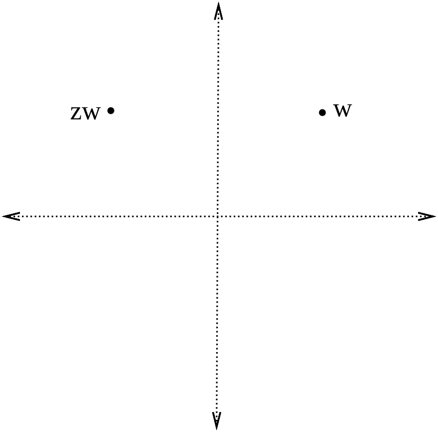


- If $G(x) = \{x\}$, then we say that x is a fixed point (G-fix).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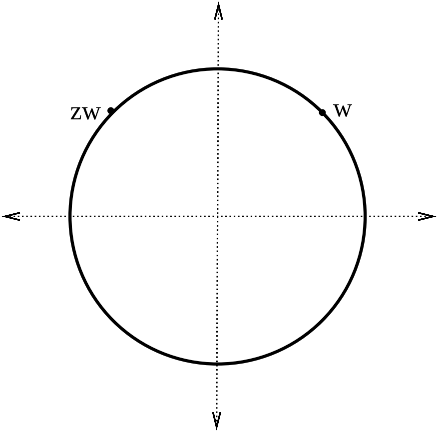
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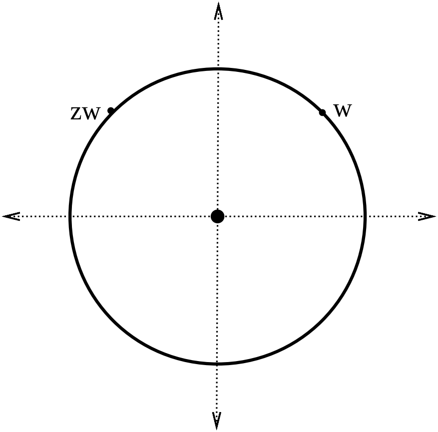
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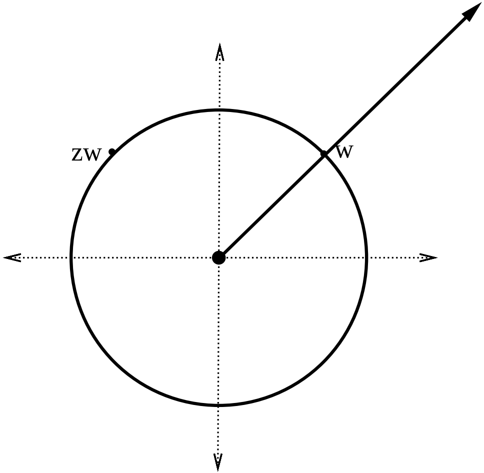
Orbit space

Definition

Let X be a G-space. Let us denote by X/G the set of all G-orbits of X. The orbit space (G-orbit space) is the set X/G equipped with the quotient topology.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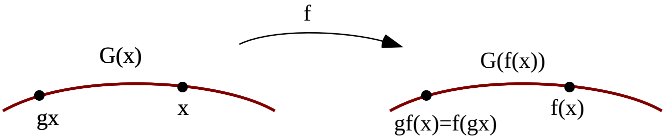
Let X and Y be G-spaces.

- A map $f : X \to Y$ is equivariant (G-equivariant), if

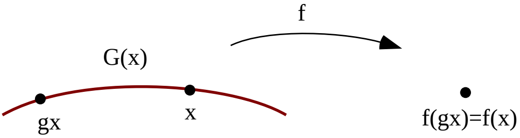
$$f(gx) = gf(x), \qquad x \in X, \quad g \in G.$$



Let X be a G-space

- A map $f : X \to Y$ is invariant if

$$f(gx) = f(x), \qquad x \in X, \quad g \in G$$



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- The map $f : \mathbb{C} \to [0, \infty)$ given by $f(z) = |z|$ is invariant.

In the hyperspace $\mathcal{B}(n)$ we can define the following action of the group $GL(n)$:

$$GL(n) \times \mathcal{B}(n) \to \mathcal{B}(n).$$

$$(g, A) \to gA = \{g(a) \mid a \in A\}.$$

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The quotient space $\mathcal{B}(n)/\sim$ becomes the orbit space

$$\mathcal{B}(n)/GL(n)$$

and therefore the Banach-Mazur compactum $BM(n)$ is homeomorphic to $\mathcal{B}(n)/GL(n)$.

In Topological Transformation Groups language, Macbeath's quotient space $cb(\mathbb{R}^n)/\sim$ is the orbit space $cb(\mathbb{R}^n)/\operatorname{Aff}(n)$, where $\operatorname{Aff}(n)$ is the group of all affine transformations of $\mathbb{R}^n$ acting by the following correspondence rule:

$$\operatorname{Aff}(n) \times cb(\mathbb{R}^n) \to cb(\mathbb{R}^n)$$

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- We will study the action of $\mathrm{Aff}(n)$ on $cb(\mathbb{R}^n)$ in order to show that $cb(\mathbb{R}^n)$ is homeomorphic to $Q \times \mathbb{R}^{n(n+3)/2}$ and $cb(\mathbb{R}^n)/Aff(n)$ is homeomorphic to the Banach-Mazur compactum $BM(n)$.

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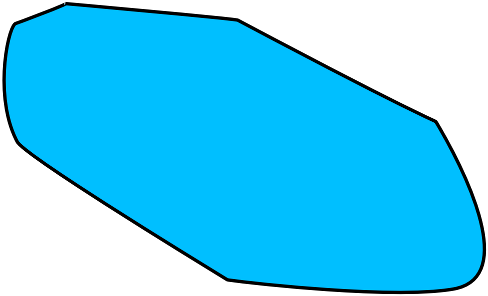
- Actions on hyperspaces of infinite dimensional spaces.

The Löwner ellipsoid

For every compact and convex body $A \in cb(\mathbb{R}^n)$ there exists a unique minimal volume ellipsoid $l(A)$ containing A. The ellipsoid $l(A)$ is usually called the Löwner ellipsoid of A.



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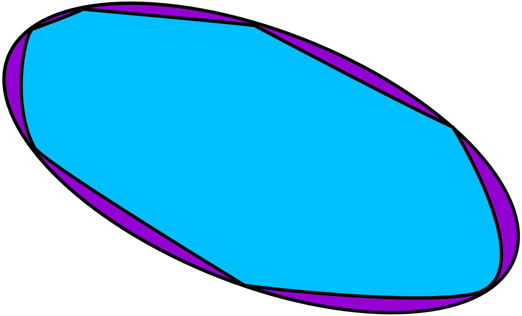
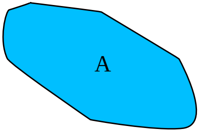


The function I is $Aff(n)-$equivariant, i.e.,

$$I(gA) = gI(A) \qquad \text{for every} \quad g \in \mathrm{Aff}(n), \text{ and } A \in cb(\mathbb{R}^n).$$



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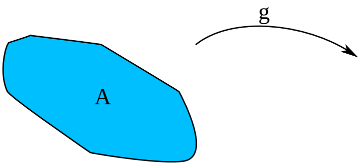

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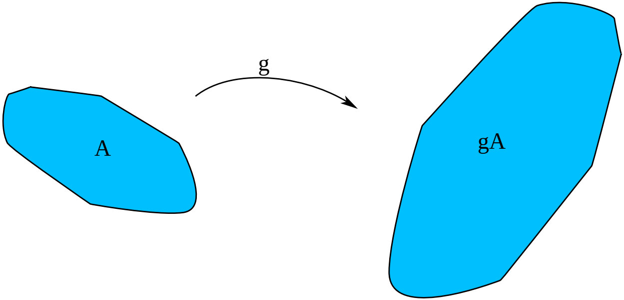

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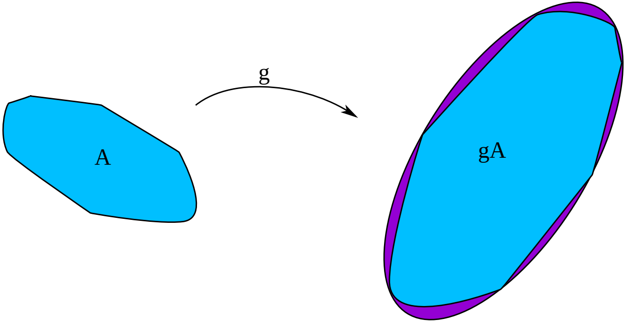

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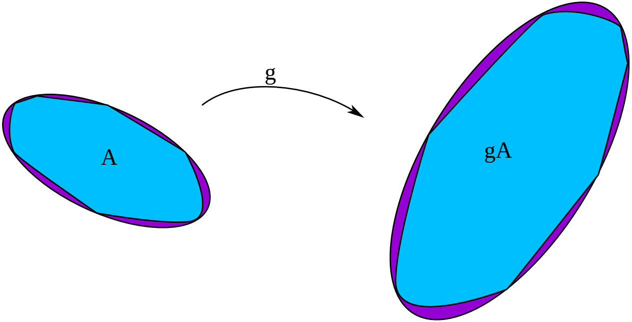

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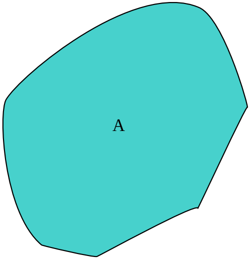

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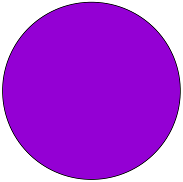

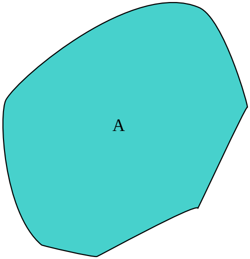

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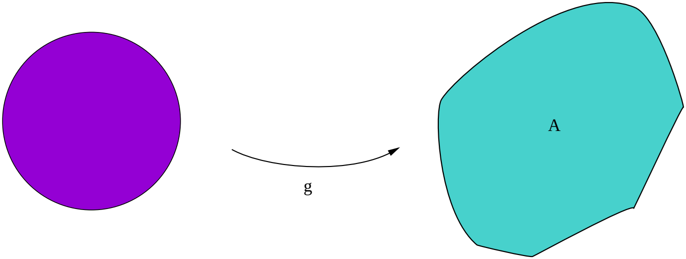

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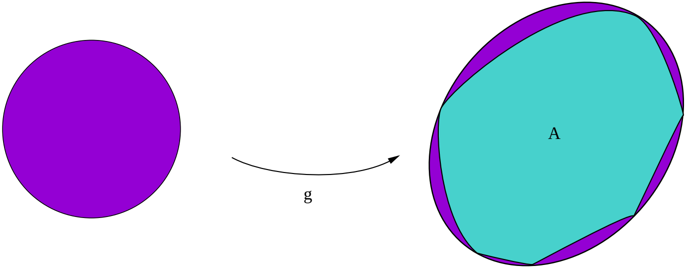

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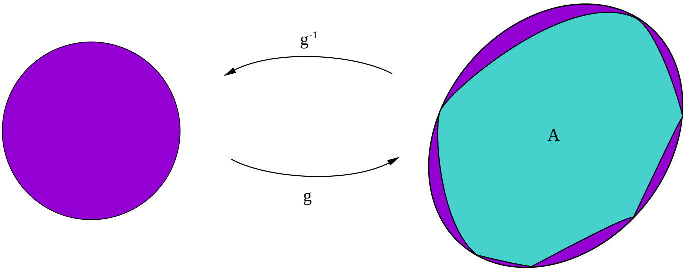

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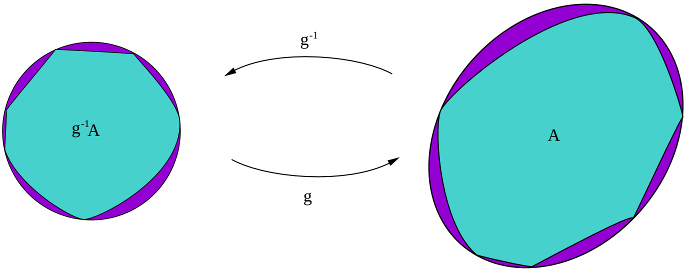

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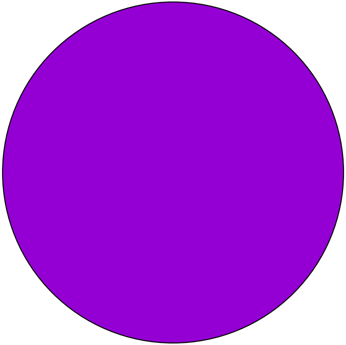
For every $n \geq 2$ lets denote by $L(n)$ the following set:

$$L(n) = \{A \in cb(\mathbb{R}^n) \mid I(A) = \mathbb{B}^n\}.$$



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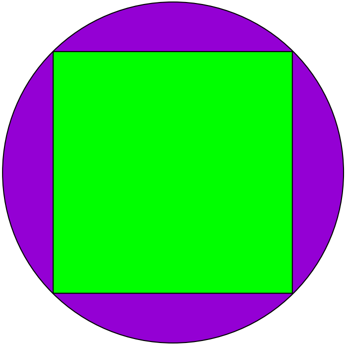

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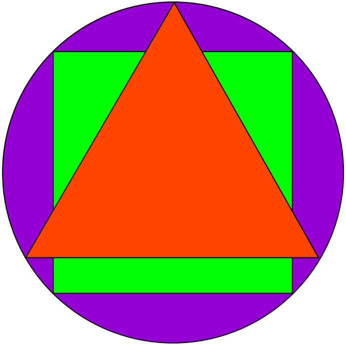

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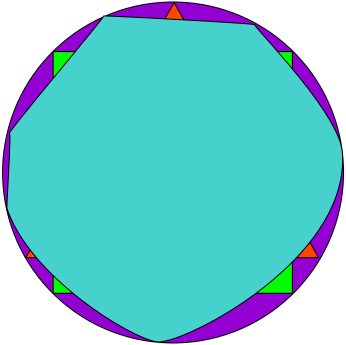

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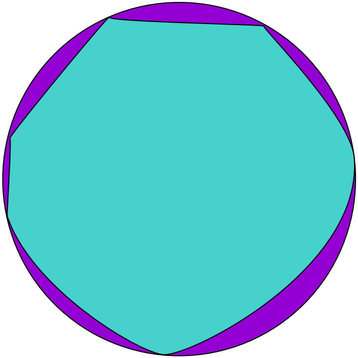

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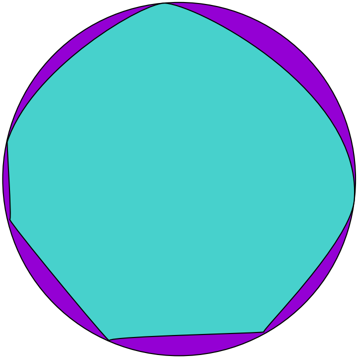

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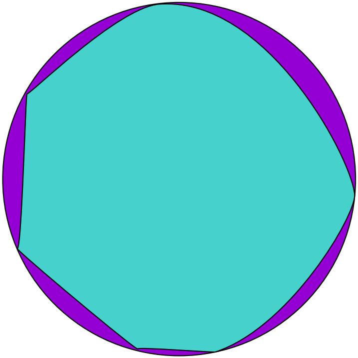

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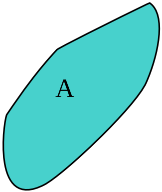
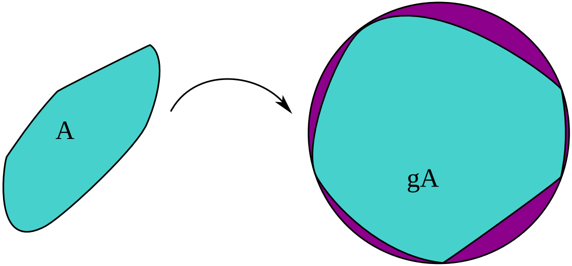
$L(n)$ is a global $O(n)$-slice


A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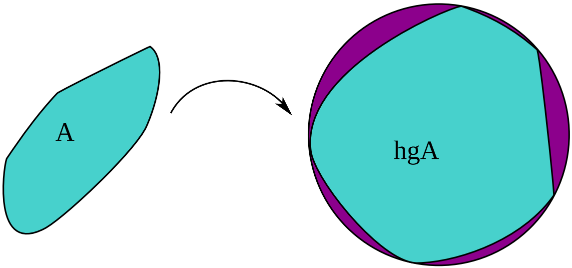
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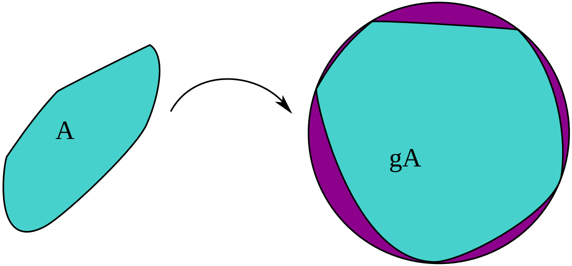
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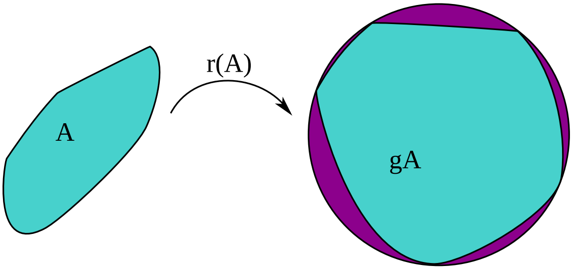
A
r(A)
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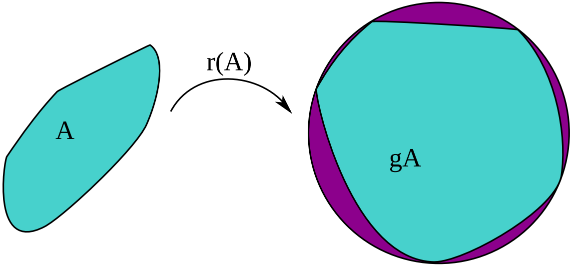

(S.A., N.J.)

There exists a continuous retraction $r : cb(\mathbb{R}^n) \to L(n)$ *such that*

- $r(A) = gA$ *for some* $g \in \mathrm{Aff}(n)$.
- r *is* $O(n)$*-invariant, i.e.,* $r(hA) = hr(A)$ *for every* $h \in O(n)$.

The induced map $\tilde{r} : cb(\mathbb{R}^n)/\operatorname{Aff}(n) \to L(n)/O(n)$ *given by:*

$$\tilde{r}(\operatorname{Aff}(n)(A)) = O(n)(r(A)).$$

is a well defined homeomorphism.

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The map $l : cb(\mathbb{R}^n) \to E(n)$ which assigns to each convex body its Löwner ellipsoid is an $\mathrm{Aff}(n)$-equivariant continuous retraction.

$$
\begin{tikzcd}
 & L(n) \\
cb(\mathbb{R}^n) \arrow[ur, "r"] \arrow[dr, "l"'] & \\
 & E(n)
\end{tikzcd}
$$

$$\begin{tikzcd}
 & L(n) & \\
cb(\mathbb{R}^n) \arrow[ur, "r"] \arrow[dr, "l"'] \arrow[rr, "r \times l"] & & L(n) \times E(n) \\
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The map $\mathrm{Aff}(n)/O(n) \to E(n)$ *given by*

$$gO(n) \to g\mathbb{B}^n$$

is a homeomorphism

Corollary

$cb(\mathbb{R}^n)$ is homeomorphic to $L(n) \times \mathrm{Aff}(n)/O(n)$.

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Question

What is $L(n)$.

(S.A., N.J.)

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A group G acts transitively on X if $G(x) = X$ for every $x \in X$.

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A Q manifold is a separable space M which is locally homeomorphic to the Hilbert cube Q.

Corollary

$cb(\mathbb{R}^n)$ is homeomorphic to $Q \times \mathbb{R}^{n(n+3)/2}$.

Theorem (S. Antonyan, 2007)

Let the orthogonal group $O(n)$ act on a Hilbert Cube X in such way that:

1. *X is an $O(n)$-AR with a unique $O(n)$-fixed point $*$,*
2. *X is strictly $O(n)$-contractible to $*$,*
3. *For a closed subgroup $K \subset O(n)$, the set $X^K = \{*\}$ if and only if K acts transitively on the unit sphere $\mathbb{S}^{n-1}$, and X^K is homeomorphic to the Hilbert cube whenever $X^K \neq \{*\}$,*
4. *For any closed subgroup $K \subset O(n)$, the K-orbit space X_0/K, is a Q-manifold, where $X_0 = X \setminus \{*\}$.*

then the $O(n)$-orbit space $X/O(n)$ is homeomorphic to the Banach-Mazur compactum $BM(n)$.

Corollary

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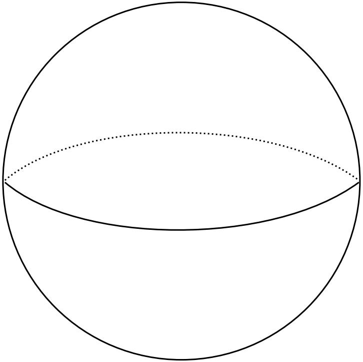
$$2^{\mathbb{S}^{n-1}} = \{A \subset \mathbb{S}^{n-1} \mid A \text{ is compact}\}$$



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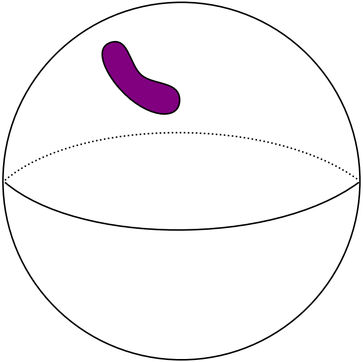

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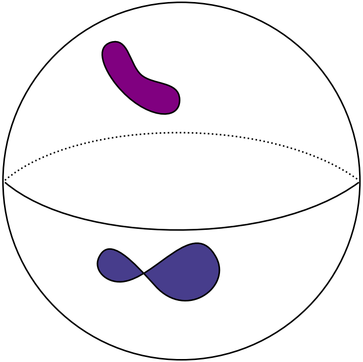

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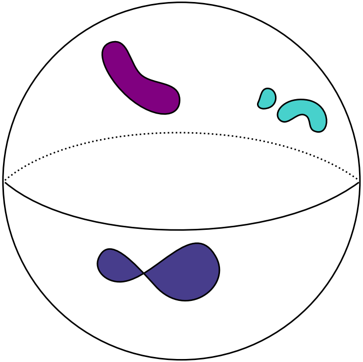

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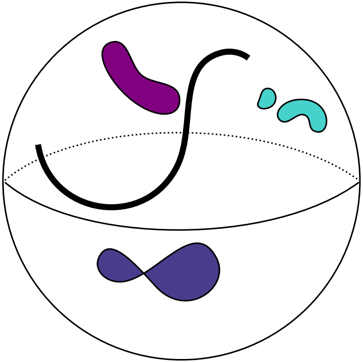

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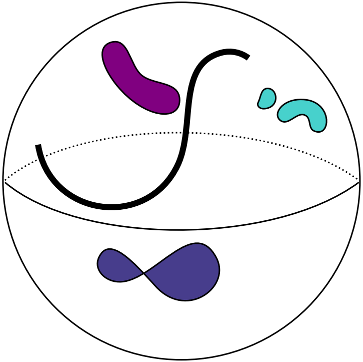
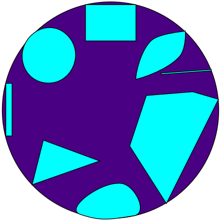


$$2^{\mathbb{S}^{n-1}}/O(n) \cong BM(n).$$

M(n)

Let us denote by $M(n)$ the subspace of $cc(\mathbb{R}^n)$ consisting of all $A \in cc(\mathbb{R}^n)$ such that

$$\max_{a \in A} \|a\| = 1.$$



$M(n)$ is the hyperspace of all compact convex subsets $A \subset \mathbb{B}^n$ such that $A \cap \mathbb{S}^{n-1} \neq \emptyset$.

Properties of $M(n)$

$M(n)$ is a $O(n)$-invariant subspace of $cc(\mathbb{R}^n)$ which satisfies the following properties:

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4. For any closed subgroup $K \subset O(n)$, the K-orbit space $M_0(n)/K$ is a Q-manifold, where $M_0(n) = M(n) \setminus \{B^n\}$
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4. For any closed subgroup $K \subset O(n)$, the K-orbit space $M_0(n)/K$ is a Q-manifold, where $M_0(n) = M(n) \setminus \{B^n\}$
5. If $K \subset O(n)$ acts nontransitively on the sphere $\mathbb{S}^{n-1}$, then the K-orbit space $M(n)/K$ is a Hilbert cube
6. $M(n)/O(n)$ is homeomorphic to the Banach-Mazur compactum $BM(n)$.

$cc(\mathbb{B}^n)$

Theorem (Nadler, Quinn, and Stavrakas (1979))

For $n \geq 2$, $cc(\mathbb{B}^n)$ is homeomorphic to the Hilbert cube $Q = \prod_{n=1}^{\infty}[-1,1]$.

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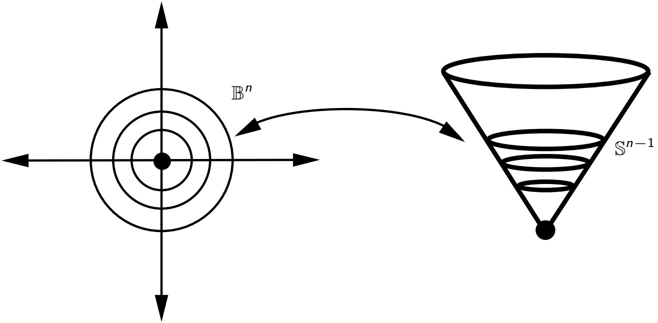
Question (Antonyan)

What is the relationship between the orbit space $cc(\mathbb{B}^n)/O(n)$ and the Banach-Mazur compactum $BM(n)$?

$\mathbb{R}^n$ is $O(n)$-homeomorphic to the cone over $\mathbb{S}^{n-1}$:

$$\mathbb{B}^n \cong_{O(n)} \mathbb{S}^{n-1} \times [0,1]/\{0\}.$$



This conic structure will induce a conic structure in $cc(\mathbb{B}^n)$. In this case, the roll of the sphere $\mathbb{S}^{n-1}$ is played by the Hyperspace $M(n)$.

Theorem

The hyperspace $cc(\mathbb{B}^n)$ is $O(n)$-homeomorphic to the cone over $M(n)$.

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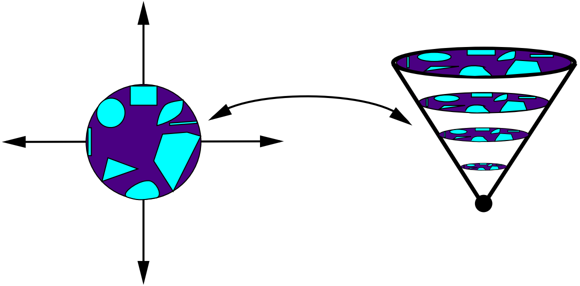

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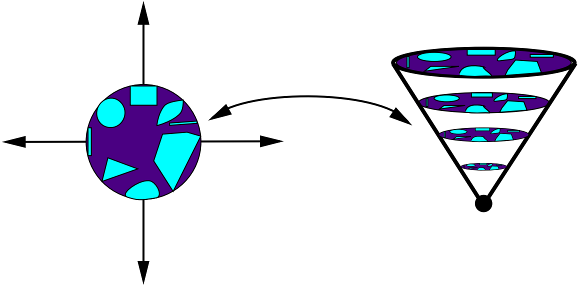

Theorem

For every closed subgroup $K \subset O(n)$, the K-orbit space $cc(\mathbb{B}^n)/K$ is homeomorphic to the cone over $M(n)/K$

Corollary

For every closed subgroup $K \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, the orbit space $cc(\mathbb{B}^n)/K$ is homeomorphic to the Hilbert cube. In particular $cc(\mathbb{B}^n)$ is homeomorphic to the Hilbert cube.

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Question

What is the relationship between the Banach-Mazur compactum $BM(n)$ and the orbit space $cc(\mathbb{B}^n)/O(n)$?

Corollary

The orbit space $cc(\mathbb{B}^n)/O(n)$ is homeomorphic to the cone over $BM(n)$.

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Question

If $n \geq 3$, Are $cc(\mathbb{B}^n)/O(n)$ and $M(n)/O(n)$ homeomorphic spaces?

Let L be a topological linear space and $K \subset L$ a convex metric subspace of L.

- Let G be a compact topological group acting on K by means of affine transformations, i.e.,

$$g(tx + (1-t)y) = tgx + (1-t)gy \quad x, y \in K, \quad g \in G.$$

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What is the topological structure of the orbit space $cc(K)/G$?

Theorem (S. Antonyan, S. Juárez, N. J)

Let L be a separable Fréchet space and let G be a compact group acting continuously and affinely on L. If the set of G-fixed points of $cc(L)$ is non locally compact, then $cc(L)/G$ is homeomorphic to the infinite dimensional separable Hilbert space ℓ_2.

Theorem (S. A, S. J, N. J.)

Let K be a Keller compactum and G be a compact group acting continuously and affinely on K. If K has a G-fixed point in the radial interior of K, then $cc(K)/G$ is homeomorphic to the Hilbert cube $Q = \prod_{n\in\mathbb{N}}[-1,1]$.

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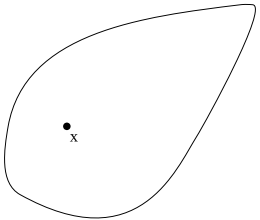
The radial interior of K is the set of all points $x \in K$ such that for every other $z \in K$ there exists $t < 0$ with $x + t(z - x) \in K$.



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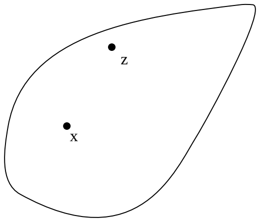

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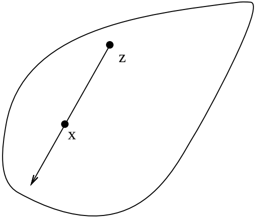

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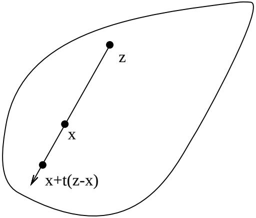

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Corollary (S. A, S. J, N. J)

Let K be a centrally symmetric Keller compactum and let G be a compact group acting continuously and affinely on K. Then the orbit space $cc(K)/G$ is homeomorphic to the Hilbert cube $Q = \prod_{n\in\mathbb{N}}[-1,1]$.

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If G is the trivial group, then $cc(K)/G \cong cc(K)$.

Corollary

If K is a Keller compactum with non-empty radial interior, then $cc(K)$ is homeomorphic to the Hilbert cube.

Proof

Theorem (Toruńczyk)

Let M be a compact metric space. M is homeomorphic to the Hilbert cube if and only if the following conditions hold:

1. *M is an AR (M is a retract of every metric space containing M as a closed subset).*
2. *For every $\varepsilon > 0$, there exist two maps $f, g : M \to M$ such that*
   - $f(M) \cap g(M) = \emptyset$
   - $d(f, 1_M) < \varepsilon$ *and* $d(g, 1_M) < \varepsilon$

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If X and Y are G-spaces and $f : X \to Y$ is G-equivariant, then the map

$$\tilde{f} : X/G \to Y/G$$

given by

$$\tilde{f}\big(G(x)\big) = G(f(x)).$$

is well defined and continuous

If (X, d) is a metric G-space and the metric is invariant $(d(gx, gy) = d(x, y))$, then

$$d^*\big(G(x), G(y)\big) = \inf\{d(x', y') \mid x' \in G(x), \quad y' \in G(y)\}$$

is a compatible metric on X/G such that

$$d^*(G(x), G(y)) \leq d(x, y)$$

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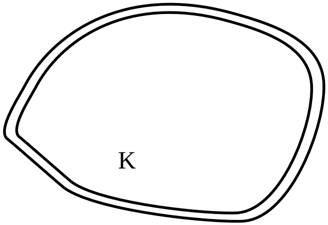
There exist a good invariant metric d on K such that d_H is G-invariant.

It is enough to construct two equivariant maps $f, g : cc(K) \to cc(K)$ satisfying:

- $f\big(cc(K)\big) \cap g\big(cc(K)\big) = \emptyset$
- $d(f, 1_{cc(K)}) < \varepsilon$ and $d(g, 1_{cc(K)}) < \varepsilon$

$$f : cc(K) \to cc(K)$$

$$f(A) = \{x \in K \mid d(x, A) \leq \varepsilon/2\}$$



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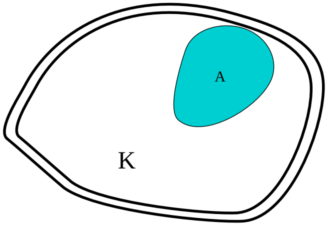

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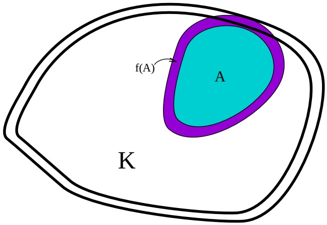

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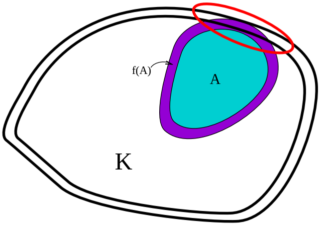
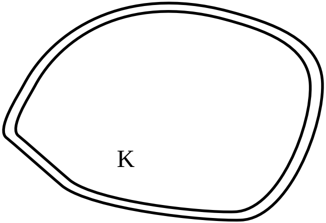


$f(A)$ intersects the radial boundary of K (the complement of the radial interior)

- Let x_0 be a G-fixed point in the radial interior of K



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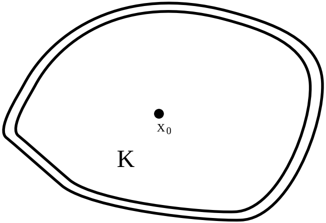

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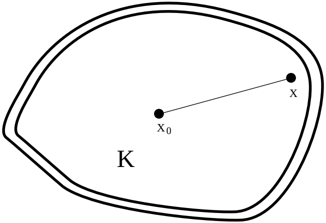

- Pick $t \in (0,1)$ such that $d(x, x_0 + t(x - x_0)) \leq \varepsilon/2$ for every $x \in K$.

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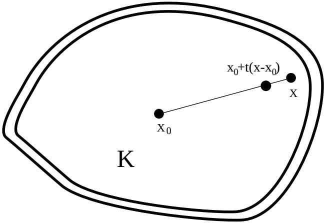

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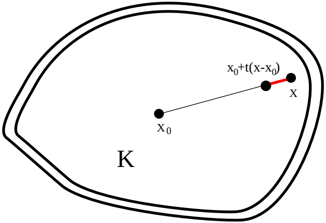
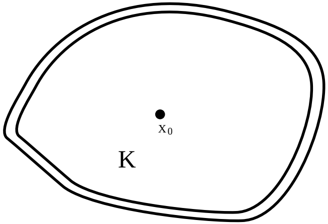


- Pick $t \in (0,1)$ such that $d(x, x_0 + t(x - x_0)) \leq \varepsilon/2$ for every $x \in K$.

$$g : cc(K) \to cc(K)$$

$$g(A) = x_0 + t(A - x_0)$$



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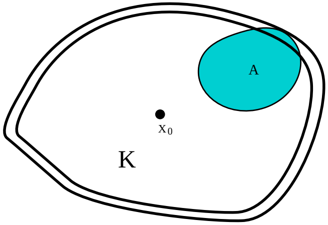

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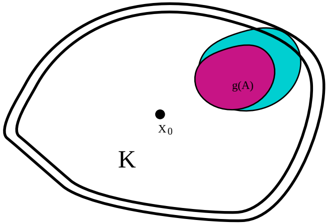

$g(A)$ is completely contained in the radial interior of K.

Final Question

If K has an empty radial interior, which is the topological structure of $cc(K)/G$?

Thank you for your attention!