Recent developments on Mizoguchi-Takahashi's fixed point theorem

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Introduction and preliminaries

Let us begin with some basic definitions and notation that will be needed in this section. Let (X, d) be metric space. For each $x \in X$ and $A \subseteq X$, let $d(x, A) = \inf_{y \in A} d(x, y)$. Denote by

- $P(X)$ the family of all nonempty subsets of X,
- $K(X)$ the family of all nonempty compact subsets of X,
- $CB(X)$ the family of all nonempty closed and bounded subsets of X.

It is clear that

$$K(X) \subseteq CB(X) \subseteq P(X).$$

- A function $H : CB(X) \times CB(X) \to [0, \infty)$ define d by

$$H(A, B) = \max\left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\}$$

  is said to be the **Pompeiu-Hausdorff metric** on $CB(X)$ induced by the metric d on X.
- An element $x \in X$ is called a fixed point of a multivalued mapping $T : X \to P(X)$ if $x \in Tx$.

It is known that many metric fixed point theorems were motivated from the celebrated Banach contraction principle which is a very powerful tool in various fields of nonlinear analysis.

Theorem (**B** (Banach contraction principle))

Let (X, d) be a complete metric space and $T : X \to X$. Assume that there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$. Then, T has a unique fixed point in X.

In 1969, Nadler [1] first gave a famous generalization of the Banach contraction principle for multivalued mapping. Since then, there has been continuous and intense research activity in multivalued mapping fixed point theory and by now there are a number of results that extend this result in many different directions.

Theorem (N (Nadler multivalued contraction principle))

Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued contraction, that is there exists $L \in [0, 1)$ such that

$$H(Tx, Ty) \leq Ld(x, y) \tag{1}$$

for all $x, y \in X$. Then, T has a fixed point in X.

[1] S.B. Nadler, Multi-valued contraction mappings, Pacific J. Math., 30 (1969), 475-488.

- One of the most important generalizations of the result of Nadler's was given by Mizoguchi and Takahashi. First, we mention about Mizoguchi-Takahashi function and features, later, we will give Mizoguchi-Takahashi's fixed point theorem [2].

- Let f be a real-valued function defined on $\mathbb{R}$. For $c \in \mathbb{R}$, we recall that

$$\limsup_{x \to c} f(x) = \inf_{\varepsilon > 0} \sup_{0 < |x - c| < \varepsilon} f(x)$$

and

$$\limsup_{x \to c^+} f(x) = \inf_{\varepsilon > 0} \sup_{0 < x - c < \varepsilon} f(x).$$

[2] N. Mizoguchi and W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl.,141 (1989), 177-188.

Definition ([3])

A function $\varphi : [0, \infty) \longrightarrow [0, 1)$ is said to be an $\mathcal{MT}$-function if it satisfies

$$\limsup_{s \longrightarrow t^+} \varphi(s) < 1$$

for all $t \in [0, \infty)$ (Mizoguchi-Takahashi's condition).

Lemma ([3])

Let $\varphi : [0, \infty) \longrightarrow [0, 1)$ be an $\mathcal{MT}$-function, then the function $\phi : [0, \infty) \longrightarrow [0, 1)$ defined as $\phi(t) = \frac{1+\varphi(t)}{2}$ is also an $\mathcal{MT}$-function.

Lemma ([3])

$\varphi : [0, \infty) \longrightarrow [0, 1)$ is an $\mathcal{MT}$-function if and only if for each $t \in [0, \infty)$, there exist $r_t \in [0, 1)$ and $\varepsilon_t > 0$ such that $\varphi(s) \leq r_t$ for all $s \in [t, t + \varepsilon_t)$.

[3] W.-S. Du, Some new results and generalizations in metric fixed point theory, Nonlinear Analysis: Theory, Methods & Applications, 73 (5) (2010), 1439-1446.

- Clearly, if $\varphi : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class. Also, $\varphi : [0, \infty) \to [0, 1)$ be defined by

$$\varphi(t) = \begin{cases} 2t & , \quad t \in \left[0, \frac{1}{2}\right) \\ 0 & , \quad \left[\frac{1}{2}, \infty\right) \end{cases}$$

is an $\mathcal{MT}$-function.

An example which is not an $\mathcal{MT}$-function is given below. Let $\varphi : [0, \infty) \to [0, 1)$ be defined by

$$\varphi(t) = \begin{cases} e^{-t} & , \quad t \neq 0 \\ 0 & , \quad t = 0 \end{cases}.$$

Since $\limsup_{s \to 0^+} \varphi(s) = 1$, φ is not an $\mathcal{MT}$-function.

We give some characterizations of $\mathcal{MT}$-functions.

Lemma [4] Let $\varphi : [0, \infty) \to [0, 1)$ be a function. Then the following statements are equivalent.

(a) φ is an $\mathcal{MT}$-function.

(b) For each $t \in [0, \infty)$, there exist $r_t^{(1)} \in [0, 1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \le r_t^{(1)}$ for all $s \in \left(t, t + \varepsilon_t^{(1)}\right)$.

(c) For each $t \in [0, \infty)$, there exist $r_t^{(2)} \in [0, 1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \le r_t^{(2)}$ for all $s \in \left[t, t + \varepsilon_t^{(2)}\right]$.

(d) For each $t \in [0, \infty)$, there exist $r_t^{(3)} \in [0, 1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \le r_t^{(3)}$ for all $s \in \left(t, t + \varepsilon_t^{(3)}\right]$.

(e) For each $t \in [0, \infty)$, there exist $r_t^{(4)} \in [0, 1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \le r_t^{(4)}$ for all $s \in \left[t, t + \varepsilon_t^{(4)}\right)$.

(f) For any nonincreasing sequence $\{x_n\} \in \mathbb{N}$ in $[0, \infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.

(g) For any strictly decreasing sequence $\{x_n\} \in \mathbb{N}$ in $[0, \infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.

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[4] W.-S. Du, On coincidence point and fixed point theorems for nonlinear multivalued maps, Topology and Its Applications, 159 (2012), 49-56.

In 1972, The following theorem, which generalizes the fixed point result for single valued mappings that was proved by Boyd and Wong [5], was proved by Reich [6]:

Theorem (R)

Let (X, d) be a complete metric space. Suppose that $T : X \to K(X)$ satisfies

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y),$$

for all $x, y \in X$, $x \neq y$, where $\alpha : (0, \infty) \to [0, 1)$ satisfies

$$\limsup_{t \to s^+} \alpha(t) < 1, \quad \textit{for all } s > 0.$$

Then T has a fixed point in X.

[5] N. Mizoguchi and W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl.,141 (1989), 177-188.

[6] S. Reich, Some fixed point problems, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur, 57 (1974), 194-198.

In 1974, Reich [7] asked that Can we take $CB(X)$ instead of $K(X)$ in Theorem R? Then, although a lot of fixed point theorist studied on this problem, it has not been completely solved. There are some partial affirmative answers to the problem and the closest answer was given by Mizoguchi and Takahashi [2], as follows:

Theorem (**MT** (Mizoguchi and Takahashi))

Let (X, d) be a complete metric space. Suppose that $T : X \to CB(X)$ satisfies

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) \tag{2}$$

for all $x, y \in X$, where φ is an $\mathcal{MT}$-function. Then T has a fixed point in X.

- In fact, the domain of φ is $(0, \infty)$ in orijinal statement. However, since $d(x, y) = 0$ implies $H(Tx, Ty) = 0$, the both are equivalent.

[2] N. Mizoguchi and W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl.,141 (1989), 177-188.

[7] S. Reich, Fixed points of contractive functions, Boll. Un. Mat. Ital., 4 (5) (1972), 26-42.

Primitive proof of Theorem MT is difficult. Another proof in [8] is not yet simple. Recently, Suzuki [9]. gave a very simple proof of Theorem MT and an example showing that it is a real generalization of Theorem N. Due to the mentioned example is complicated, here we consider another simple example as follows:

Example

Let $X = [0, \infty)$ and

$$d(x,y) = \begin{cases} \max\{x,y\} & , \quad x \neq y \\ 0 & , \quad x = y \end{cases},$$

then (X, d) is complete metric space. Let $T : X \to CB(X)$ be defined by

$$Tx = \left[0, \frac{x^2}{x+1}\right].$$

and a function $\varphi : [0,\infty) \to [0,1)$ be defined by $\varphi(t) = \frac{t}{t+1}$. It is obvious that φ is an $\mathcal{MT}$-function.Then, the all condition of Theorem MT is satisfied.

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[8] P. Z. Daffer and H. Kaneko, Fixed points of generalized contractive multivalued mappings, J. Math. Anal. Appl., 192 (1995), 655-666.

[9] T. Suzuki, Mizoguchi-Takahashi's fixed point theorem is a real generalization of Nadler's, J. Math. Anal. Appl., 340 (2008), 752-755.

Example (continued)

In fact, for $x > y$ with $x \neq y$, we have

$$H(Tx, Ty) = \frac{x^2}{x+1} \leq \frac{x}{x+1}.x = \varphi(d(x,y))d(x,y).$$

Note that if $x = y$, (2) is clearly satisfied. Thus all conditions of Theorem MT are satisfied and so T has a fixed point in X.
On the other hand, it is easy to show that Theorem N is not applicable in this case.
Indeed, assume there exists $L \in [0,1)$ such that (1) holds true, then
$H(Tx, T0) = \frac{x^2}{x+1} \leq Lx$, for all $x \geq 0$. This implies

$$\lim_{x\to\infty} \frac{H(Tx, T0)}{d(x,0)} = \lim_{x\to\infty} \frac{\frac{x^2}{x+1}}{x} = 1 \leq L,$$

which a contradiction.

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In 2007, M. Berinde and V. Berinde [10] proved the following interesting fixed point theorem.

Theorem (BB (M. Berinde and V. Berinde))

Let (X, d) be a complete metric space, $T : X \to CB(X)$. Suppose that there exist a constant $L \geq 0$ such that

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(y, Tx)$$

for all $x, y \in X$, where φ is an $\mathcal{MT}$-function. Then T has a fixed point in X.

It is clear that if $L = 0$ in Theorem BB, then we can obtain Theorem MT.

[10] M. Berinde and V. Berinde, On a general class of multi-valued weakly Picard mappings, Journal of Mathematical Analysis and Applications, 326 (2007), 772-782.

In 2012, Samet et al. [11] were first to introduce the concept of α-ψ-contractive and α admissible mapping self-mappings and they proved some the interesting fixed point results for such mappings on complete metric spaces (See [12, 13, 14]). They also gave some examples and applications to ordinary differential equations of the obtained results. Asl et al [15] characterized these notions to multivalued mappings by introducing the notions of α_*-ψ-contractive and α_*-admissible mappings and obtaind some fixed-point results for multivalued mappings.

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[11] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Analysis 75 (2012), 2154-2165.

[12] E. Karapınar and B. Samet, Generalized α-ψ-contractive type mappings and related fixed point theorems with applications, Abstract and Applied Analysis 2012 (2012), Article ID 793486, 17 pages.

[13] B. Mohammadi, S. Rezapour and N. Shahzad, Some results on fixed points of α-ψ-Ćirić generalized multifunctions, Fixed Point Theory and Applications Article ID 24 (2013), 10 pages.

[14] H. Nawab, E. Karapınar, P. Salimi and F. Akbar, α-admissible mappings and related fixed point theorems, Journal of Inequalities and Applications 114 (2013), 11 pages.

[15] J.H. Asl, S. Rezapour and N. Shahzad, On fixed points of α-ψ-contractive multifunctions, Fixed Point Theory and Applications 212 (2012), 6 pages, doi:10.1186/1687-1812-2012-212.

Now, we recall these definitions and results. Let (X, d) be a metric space, $T : X \to P(X)$ and $\alpha : X \times X \to [0, \infty)$ be a function. Then, we say that:

- T is an α-admissible mapping whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$ implies $\alpha(y, z) \geq 1$ for all $z \in Ty$,
- T is an α_*-admissible mapping whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$ implies $\alpha_*(Tx, Ty) \geq 1$, where $\alpha_*(Tx, Ty) = \inf\{\alpha(a, b) : a \in Tx, b \in Ty\}$,
- α has (B) property whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

It is easy to see that α_*-admissible mapping are also α-admissible mapping, but the converse may not be true as shown in Example 15 of [16] as follows:

Example

Let $X = [-1, 1]$ and $\alpha : X \times X \to [0, \infty)$ is defined by

$$\alpha(x, y) = \begin{cases} 0 & , \quad x = y \\ \\ 1 & , \quad x \neq y \end{cases}.$$

Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \{-x\} & , \quad x \notin \{-1, 0\} \\ \{0, 1\} & , \quad x = -1 \\ \{1\} & , \quad x = 0 \end{cases}.$$

[16] G. Mınak, Ö. Acar and I. Altun, Multivalued pseudo-Picard operators and fixed point results, Journal of Function spaces and applications, 2013 (2013), Article ID 827458, 7 pages.

Example (continued)

Let $x = -1$, and $y = 0 \in Tx = \{0, 1\}$, then $\alpha(x, y) \geq 1$, but

$$\alpha_*(Tx, Ty) = \alpha_*(\{0, 1\}, \{1\}) = 0.$$

Thus T is not α_*-admissible. Now we show that, T is α-admissible with the following cases:

Case 1. If $x = 0$, then $y = 1$ and $\alpha(x, y) \geq 1$. Also, $\alpha(y, z) \geq 1$ since $z = -1 \in Ty = \{-1\}$.

Case 2. If $x = -1$, then $y \in \{0, 1\}$ and $\alpha(x, y) \geq 1$. Also $\alpha(y, z) \geq 1$ for all $z \in Ty$.

Case 3. If $x \notin \{-1, 0\}$, then $y = -x$ and $\alpha(x, y) \geq 1$. Also $\alpha(y, z) \geq 1$ since $z = x \in Ty = \{x\}$.

- Let Ψ be the family of nondecreasing functions

$$\psi : [0,\infty) \to [0,\infty) \text{ such that } \sum_{n=1}^{\infty} \psi^n(t) < \infty \text{ for all } t > 0,$$

  where ψ^n is the n th iterate of ψ. It is easily proved that if $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.

- Let (X, d) be a metric space and $\psi \in \Psi$. A multivalued mapping $T : X \to CB(X)$ is called multivalued α-ψ-contractive whenever for all $x, y \in X$

$$\alpha(x,y)H(Tx,Ty) \leq \psi(d((x,y)),$$

  and multivalued α_*-ψ-contractive whenever for all $x, y \in X$

$$\alpha_*(Tx,Ty)H(Tx,Ty) \leq \psi(d((x,y)).$$

The fixed point results for these type mappings are given as follows:

Theorem (**MRS [13]**)

Let (X, d) be a complete metric space, $\psi \in \Psi$ be a strictly increasing mapping and $T : X \to CB(X)$ be an α-admissible and multivalued α-ψ-contractive on X. Suppose there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is continuous or α has (B) property, then, T has a fixed point in X.

Theorem (**ARS [15]**)

Let (X, d) be a complete metric space, $\psi \in \Psi$ be a strictly increasing mapping and $T : X \to CB(X)$ be an α_*-admissible and multivalued α_*-ψ-contractive on X. Suppose there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is continuous or α has (B) property, then, T has a fixed point in X.

[13] B. Mohammadi, S. Rezapour and N. Shahzad, Some results on fixed points of α-ψ-Ćirić generalized multifunctions, Fixed Point Theory and Applications Article ID 24 (2013), 10 pages.

[15] J.H. Asl, S. Rezapour and N. Shahzad, On fixed points of α-ψ-contractive multifunctions, Fixed Point Theory and Applications 212 (2012), 6 pages, doi:10.1186/1687-1812-2012-212.

Mınak and Altun present some generalizations of Teorem MT using this new idea, as follows:

Theorem (MA1 [17])

Let (X, d) be a complete metric space and $T : X \to CB(X)$ be an α-admissible multivalued mapping such that

$$\alpha(x, y)H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) \tag{3}$$

for all $x, y \in X$, where φ is an $\mathcal{MT}$-function. Suppose there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is continuous or α has (B) property, then T has a fixed point in X.

[17] G. Mınak and I. Altun, Some new generalizations of Mizoguchi-Takahashi type fixed point theorem, Journal of Inequalities and Applications, 2013, 2013:493.

Although α_*-admissibility implies α-admissibility of T, we will give the following theorem. Because, the contractive condition is slight different from (3).

Theorem (**MA2** [17])

Let (X, d) be a complete metric space and $T : X \to CB(X)$ be an α_*-admissible multivalued mapping such that

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \varphi(d(x, y))d(x, y)$$

for all $x, y \in X$, where φ is an $\mathcal{MT}$-function. Suppose there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is continuous or α has (B) property, then T has a fixed point in X.

[17] G. Mınak and I. Altun, Some new generalizations of Mizoguchi-Takahashi type fixed point theorem, Journal of Inequalities and Applications, 2013, 2013:493.

Now we give an example to illustrate our result. Note that TheoremMT can not be applied to this example.

Example ([17])

Let $X = [-1, 1]$ and $d(x, y) = |x - y|$. Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \{2x+1\} & , \quad x \in \left[-1, -\frac{3}{4}\right) \\ \{2x-1\} & , \quad x \in \left(\frac{3}{4}, 1\right] \\ \left[-\frac{1}{2}, \frac{1}{2}\right] & , \quad x \in \left[-\frac{3}{4}, \frac{3}{4}\right] \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & , \quad x, y \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0 & , \quad \text{otherwise} \end{cases}.$$

[17] G. Mınak and I. Altun, Some new generalizations of Mizoguchi-Takahashi type fixed point theorem, Journal of Inequalities and Applications, 2013, 2013:493.

Example (continued)

Then T is an α_*-admissible and

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) \tag{4}$$

for all $x, y \in X$, where φ is any $\mathcal{MT}$-function.

Indeed, first, we show that T is an α_*-admissible. If $\alpha(x, y) \geq 1$, then $x, y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and hence

$$\begin{aligned} \alpha_*(Tx, Ty) &= \alpha_*\left(\left[-\frac{1}{2}, \frac{1}{2}\right], \left[-\frac{1}{2}, \frac{1}{2}\right]\right) \\ &= \inf\left\{\alpha\left(a, b\right) : a, b \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\} \\ &= 1. \end{aligned}$$

Therefore T is an α_*-admissible.

Example (continued)

Now we consider the following cases:

Case 1. Let $x, y \in X$ with $\{x, y\} \cap \{[-1, -\frac{3}{4}) \cup (\frac{3}{4}, 1]\} \neq \emptyset$, then $\alpha_*(Tx, Ty) = 0$. Thus (4) is satisfied.

Case 2. Let $x, y \in X$ with $x, y \in [-\frac{3}{4}, \frac{3}{4}]$, then

$$\begin{aligned} H(Tx, Ty) &= H\left(\left[-\frac{1}{2}, \frac{1}{2}\right], \left[-\frac{1}{2}, \frac{1}{2}\right]\right) \\ &= 0 \end{aligned}$$

and so again (4) is satisfied.

Now, if $x, y \in (\frac{3}{4}, 1]$ with $x \neq y$ we have

$$\begin{aligned} H(Tx, Ty) &= H(\{2x - 1\}, \{2y - 1\}) \\ &= 2d(x, y). \end{aligned}$$

Therefore there is no any $\mathcal{MT}$-function satisfying Theorem MT.

- If we take $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$, then any multivalued mappings $T : X \to CB(X)$ are α-admissible as well as α_*-admissible. Therefore, Theorem MT is a special case of Theorem MA1 and Theorem MA2.

THANK YOU FOR YOUR ATTENTION