Frames in Fréchet spaces

FNRS Group - Functional Analysis

Esneux

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Bases and their limitations

The main point is the missing *flexibility*. The conditions for being a basis are so strong that:

- It is often impossible to construct bases with special properties;
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A basis is characterized by the expansion property; this is, a basis $\{e_j\}_j$ for a normed vector space E allows us to represent every $x \in E$ as a (maybe infinite) *unique* linear combination of the basis elements,

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with coefficients $\{c_j\}$ that depend linearly on x.
One might ask whether uniqueness is really needed?

Non-bases with the expansion property

Let $\{e_k\}_{k\in\mathbb{Z}} = \{e^{2\pi ikx}\}_{k\in\mathbb{Z}}$ be the orthonormal basis for the Hilbert space $L^2[0,1]$ with the following inner product associated
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We consider $I \subsetneq [0,1]$ a proper subinterval and we identify $L^2(I)$ as a subspace of $L^2[0,1]$ such that the functions are zero in $[0,1]\backslash I$.

A function $f \in L^2(I)$ is identified as a function $f \in L^2[0,1]$ such that $f = \sum_{k\in\mathbb{Z}}\langle f, e_k\rangle e_k$ in $L^2[0,1]$. We also have that $f = \sum_{k\in\mathbb{Z}}\langle f, e_k\rangle e_k$ en $L^2(I)$.

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However, $\{e_k\}_{k\in\mathbb{Z}}$ are not a basis for $L^2(I)$.

Non-bases with the expansion property

To see this, we define the function:

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{, if } x \in I \\ 1 & \text{, if } x \notin I \end{cases} \quad \text{then } \widetilde{f} = \sum_{k\in\mathbb{Z}} \langle \widetilde{f}, e_k \rangle e_k \text{ in } L^2[0,1].$$

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By restricting to I, the expansion is also valid in $L^2(I)$; since $f = \widetilde{f}$ on I, this shows that $f = \sum_{k\in\mathbb{Z}} \langle \widetilde{f}, e_k \rangle e_k$ in $L^2(I)$.

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Thus are both non-identical expansions of f in $L^2(I)$ such that $\{\langle f, e_k \rangle\}_{k\in\mathbb{Z}} \neq \{\langle \widetilde{f}, e_k \rangle\}_{k\in\mathbb{Z}}$.

Frame theory applications

Frames theory is a research area in mathematics, computer science and engineering applied in many different fields as:

- Sampling theory.
- Speech processing.
- Biomedical signal processing.
- Wavelet theory.
- Time-frequency analysis. Applications in image and signal processing.

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In particular, having more elements than needed for a basis turns out to have a certain noise suprissing effect.

Aim

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Schauder frames have been investigated by **Casazza, Gröchenig, Carando, Lassalle, Schmidberg, Korobeĭnik, Taskinen and others**.

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Let $\{x_j\}_{j=1}^{\infty} \subset E$ and let $\{x_j'\}_{j=1}^{\infty} \subset E'$, we say that $(\{x_j'\}, \{x_j\})$ is a *Schauder frame of E* if

$$x = \sum_{j=1}^{\infty} x_j'(x)\, x_j, \qquad \text{for all } x \in E,$$

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the series converging in E.

We denote by ω the space $\mathbb{K}^{\mathbb{N}}$ endowed by the product topology.
A sequence space is a lcs Λ such that $\mathbb{K}^{(\mathbb{N})} \subset \Lambda \subset \omega$, this last inclusion being continuous.

Examples

Example (Leont'ev, 1970's)

For every convex bounded set, $\Omega \subset \mathbb{C}$, there exists a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{C}$ such that, for every $f \in \mathcal{H}(\Omega)$,

$$f(z) = \sum_{j=1}^{\infty} c_j e^{x_j z}$$

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Korobeĭnik, Y. F. and Melikhov, S. N. proved that, if the boundary of Ω is C^2, there exist $\{c_j\}_{j=1}^{\infty}$ depending continuously of f (i.e. $c_j := u_j(f)$ where u_j is a linear and continuous operator). Therefore, we obtain a Schauder frame.

Examples

Example

Let E be a lcs with a Schauder basis $\{e_j\}_{j=1}^{\infty} \subset E$ and denote by $\{e_j'\}_{j=1}^{\infty} \subset E'$ the functional coefficients. Then $(\{e_j'\}, \{e_j\})$ is a Schauder frame for E such that $e_j'(e_i) = \delta_{j,i}$ for all $i, j \in \mathbb{N}$.

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Example

Let E be a lcs and let $P : E \to E$ be a continuous linear projection. If $(\{x'_j\}, \{x_j\})$ is a Schauder frame for E, then $(\{P'(x'_j)\}, \{P(x_j)\})$ is a Schauder frame for $P(E)$.

Theorem

From now on E always be a barrelled and complete Hausdorff locally convex space.

Barrelled locally convex spaces are those satisfying the uniform boundedness principle (Banach-Steinhaus' Theorem).

Example

Fréchet spaces (complete and metrizable locally convex spaces).

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Theorem

The following are equivalent:

1. *E admits a Schauder frame.*
2. *E is isomorphic to a complemented subspace of a complete sequence space with Schauder basis.*

Sketch of the Proof

1) $\Rightarrow$ 2)

- We define an injective and continuous linear map

$$\begin{array}{rcl} U : E & \longrightarrow & \Lambda \\ x & \longrightarrow & U(x) := \{x_j'(x)\}_j. \end{array}$$

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$$\begin{array}{rcl} U : E & \longrightarrow & \bigwedge \\ x & \longrightarrow & U(x) := \{x_j'(x)\}_j. \end{array}$$

Where $\bigwedge := \{\alpha = \{\alpha_j\}_j \in \omega : \sum_{j=1}^{\infty} \alpha_j x_j$ is convergent in $E\}$ is a sequence space endowed with the system of seminorms

$$\mathcal{Q} := \{q_p(\{\alpha_j\}_j) := \sup_n p(\sum_{j=1}^{n} \alpha_j x_j), \text{ for all } p \in cs(E)\},$$

such that $(\bigwedge, \mathcal{Q})$ is complete.

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- And $S : \bigwedge \longrightarrow E,\ S(\{\alpha_j\}_j) := \sum_{j=1}^{\infty} \alpha_j x_j$, is a continuous linear map.
- From $S \circ U = I_E$ we conclude that U is an isomorphism into its range $U(E)$ and $U \circ S$ is a projection of $\bigwedge$ onto $U(E)$.

Bounded approximation property

Definition

A lcs E has the *bounded approximation property* (BAP) if there exists $\{A_j\}_{j\in J} \subset L(E,E)$ a net which is equicontinuous with $\dim(A_j(E)) < \infty$ for every $j \in J$ and $\lim_{j\in J} A_j(x) = x$ for every $x \in E$. In other words, $A_j \to I$ in $L_s(E)$.

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Theorem (Pełczyński)

Let E be a separable Fréchet space, the following are equivalent:

1. The bounded approximation property holds in E.

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Theorem (Pełczyński)

Let E be a separable Fréchet space, the following are equivalent:

1. The bounded approximation property holds in E.
2. E is isomorphic to a complemented subspace of a complete sequence space with Schauder basis.
3. E admits a Schauder frame.

Bounded approximation property

Dubinsky (1981), Vogt (2010) gave examples of nuclear (hence separable) Fréchet spaces E which do not have the bounded approximation property. These separable Fréchet spaces E do not admit a Schauder frame.

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Although by Kōmura-Kōmura's Theorem they are isomorphic to a subspace of the countable product $s^{\mathbb{N}}$ of copies of the space of rapidly decreasing sequence, that has a Schauder basis.

Perturbation Result

Theorem

Let $(\{x_j'\}, \{x_j\})$ be a Schauder frame of a complete lcs E. Then, if $\{y_j\}_{j=1}^{\infty}$ is a sequence in E satisfying that $\exists p_0 \in cs(E)$ such that for all $p \in cs(E)$ there is $C_p > 0$ with:

(i) $\sum_{j=1}^{\infty} |x_j'(x)| p(x_j - y_j) \leq p_0(x) C_p$ for each $x \in E$ and

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then, there exists $\{y_j'\}_{j=1}^{\infty}$ a sequence in E' such that $(\{y_j'\},\{y_j\})$ is a Schauder frame for E.

It follows from a result of *Garnir, De Wilde, Schmets* related to the condition of invertibility of the operator $I - T$ given an operator T.

Example

Proposition

Let $(\{x_j'\}, \{x_j\})$ be a Schauder frame for a barrelled lcs E and let $\{x_j\}_j$ being bounded below (i.e. there exists $p \in cs(E)$ such that $p(x_j) \geq 1$ for every $j \in \mathbb{N}$). Then, $\{x_j'\}_j$ is equicontinuous in E'.

Example

Considering the system of seminorms $\{q_n\}_n$ for $C^\infty(K)$ given by $q_n(f) := \sup\{|f^{(\alpha)}(x)| : x \in K, \, |\alpha| \leq n\}, n \in \mathbb{N}_0$.

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By previous result, observe that, if $x_1'(x_1) \neq 1$ the map $x \to \sum_{j=2}^{\infty} x_j'(x)x_j$ is invertible as 1 is not an eigenvalue of the rank one operator $x \to x_1'(x)x_1$.

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Hence there exists $\{y_j'\}_j \subset E'$ such that $(\{y_j'\}_j, \{x_{j+1}\}_j)$ is a Schauder frame.

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Hence there exists $\{y'_j\}_j \subset E'$ such that $(\{y'_j\}_j, \{x_{j+1}\}_j)$ is a Schauder frame.

In that case, we can remove an element and still obtain Schauder frames.

Introduction

Given a Schauder frame $(\{x_j'\}, \{x_j\})$ of E it is rather natural to ask whether $(\{x_j\}, \{x_j'\})$ is a Schauder frame of E'.

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Lemma

If $(\{x_j'\}, \{x_j\})$ is a Schauder frame of E, then $(\{x_j\}, \{x_j'\})$ is a Schauder frame of $(E', \sigma(E', E))$.

Introduction

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The question we are going to face is under which conditions $(\{x_j\}, \{x_j'\})$ is a Schauder frame of $(E', \beta(E', E))$.

Shrinking Schauder frames

We define a linear operator $T_n : E \to E$ as $T_n(x) := \sum_{j=n+1}^{\infty} x_j'(x) x_j$.

Definition

An Schauder frame $(\{x_j'\}, \{x_j\})$ is *shrinking* if for all $x' \in E'$, $\lim_{n\to\infty} x' \circ T_n = 0$ uniformly on the bounded subsets of E.

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The following are equivalent:

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The following are equivalent:

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2. $(\{x_j\}, \{x'_j\})$ is a Schauder frame for E'_β.
3. For all $x' \in E'$, $\sum_{j=1}^{\infty} x'(x_j)\, x'_j$ is convergent in E'_β.

Montel spaces

Definition

A space E is called Montel if it is barrelled and every bounded subset of E is relatively compact.

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Since the pointwise convergence of an equicontinuous sequence of operators implies the uniform convergence on the compact sets.

Montel spaces

Beanland, Freeman, Liu, 2012

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The idea of the proof is the existence of weak* null sequences in the unit sphere of E'.

Theorem

Let E be a separable Fréchet space with the bounded approximation property. Then E is Montel if and only if every Schauder frame of E is shrinking.

Using *Bonet, Lindstrom, Valdivia (1993)* a Fréchet space E is Montel if and only if every weak* null sequence in E' is also strongly convergent.

Boundedly Complete Schauder frames

Definition

A Schauder frame $(\{x_j'\}, \{x_j\})$ is *boundedly complete* if for all $x'' \in E''_\beta$, the series $\sum_{j=1}^{\infty} x_j'(x'')\, x_j$ converges in E.

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Proposition

1. Let $\{e_j\}_{j=1}^{\infty}$ be a Schauder basis of E; $(\{e_j'\}, \{e_j\})$ is a boundedly complete Schauder frame if and only if $\{e_j\}_{j=1}^{\infty}$ is a boundedly complete Schauder basis (i.e. for every $\{\alpha_j\}_{j=1}^{\infty} \subset \mathbb{K}$ such that $(\sum_{j=1}^{k} \alpha_j e_j)_k$ is bounded, then $\sum_{j=1}^{\infty} \alpha_j e_j$ is convergent).

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2. If $(\{x_j'\}, \{x_j\})$ is a boundedly complete Schauder frame for E with E''_β barrelled, then E is complemented in its bidual E''_β.

Properties

Proposition

Let E be a lcs and let $(\{x_j'\}, \{x_j\})$ be a shrinking Schauder frame of E. Then $(\{x_j\}, \{x_j'\})$ is a boundedly complete Schauder frame of E_β'.

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Let E be a lcs and let $(\{x_j'\}, \{x_j\})$ be a shrinking Schauder frame of E. Then $(\{x_j\}, \{x_j'\})$ is a boundedly complete Schauder frame of E_β'.

Theorem

Let $(\{x_j'\}, \{x_j\})$ be a Schauder frame of a (barrelled) lcs E which is shrinking and boundedly complete, then E is (semi-)reflexive.

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Theorem

Let $(\{x_j'\}, \{x_j\})$ be a Schauder frame of a (barrelled) lcs E which is shrinking and boundedly complete, then E is (semi-)reflexive.

Fix $x'' \in E''$.

$$\langle x'', x' \rangle = \langle x'', \sum_{j=1}^{\infty} x'(x_j) x_j' \rangle = \sum_{j=1}^{\infty} x'(x_j) x''(x_j') = (\sum_{j=1}^{\infty} x''(x_j') x_j)(x') = \langle x, x' \rangle.$$

Definition

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An Schauder frame $(\{x_j'\}, \{x_j\})$ for a lcs E is said to be *unconditional* if for every $x \in E$ we have $x = \sum_{j=1}^{\infty} x_j'(x)\, x_j$ with unconditional convergence.

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McArthur, Retherford, 1969

If a series $\sum_{j=1}^{\infty} x_j$ converges unconditionally, then, for every bounded sequence of scalars $\{a_j\}$, the series $\sum_{j=1}^{\infty} a_j x_j$ converges and the operator

$$\begin{array}{ccc} \ell_\infty & \longrightarrow & E \\ \{a_j\} & \longrightarrow & \sum_{j=1}^{\infty} a_j x_j; \end{array}$$

is a continuous linear operator.

Space structure

Let $\{x_j\}_{j=1}^{\infty} \subset E$ be a sequence of non-zero elements.

$$\widetilde{\Lambda} := \{\alpha = \{\alpha_j\}_j \in \omega : \sum_{j=1}^{\infty} \alpha_j x_j \text{ is convergent in } E\} \subset \omega,$$

$$\widetilde{\mathcal{Q}} := \{q_p(\{\alpha_j\}_j) := \sup_n p(\sum_{j=1}^{n} \alpha_j x_j), \text{ for all } p \in \mathcal{P}\}.$$

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Proposition

If E is a complete lcs then $(\widetilde{\Lambda}, \widetilde{\mathcal{Q}})$ is a complete lcs.

Space structure

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Proposition

If E is a complete lcs then $(\widetilde{\bigwedge}, \widetilde{\mathcal{Q}})$ is a complete lcs.

Theorem

If $(\{x_j'\}, \{x_j\})$ is a unconditional Schauder frame of E, then is isomorphic to a complemented subspace of $(\widetilde{\bigwedge}, \widetilde{\mathcal{Q}})$.

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The following are equivalent:

1. E admits an unconditional Schauder frame.

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The following are equivalent:

1. *E admits an unconditional Schauder frame.*
2. *E is isomorphic to a complemented subspace of a complete sequence space with unconditional Schauder basis.*

Boundedly retractive (LF)-spaces

Definition

An (LF)-space $E = \mathrm{ind}_{n\rightarrow} E_n$ is called *boundedly retractive* if for every bounded set B in E there exists $m = m(B)$ such that B is contained and bounded in E_m and E_m and E induce the same topology on B.

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By *Fernández (1990)* an (LF)-space E is boundedly retractive if and only if each bounded subset in E is in fact bounded in some step E_n and for each n there is $m > n$ such that E_m and E induce the same topology on the bounded sets of E_n.

Wengenroth (1996) also proved an equivalence between boundedly retractive (LF)-spaces and Retakh's condition (M) for arbitrary (LF)-spaces.

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Wengenroth (1996) also proved an equivalence between boundedly retractive (LF)-spaces and Retakh's condition (M) for arbitrary (LF)-spaces.

Each Fréchet space F can be seen as a boundedly retractive (LF)-space, just take $F_n = F$ for all $n \in \mathbb{N}$. In particular the following result holds for Fréchet spaces.

Rosenthal's ℓ_1 theorem for (LF)-spaces

Rosenthal's ℓ_1 theorem was extended to Fréchet spaces by *Díaz (1989)*, showing that every bounded sequence in a Fréchet space has a subsequence that is either weakly Cauchy or equivalent to the unit vectors in ℓ_1.

Rosenthal's ℓ_1 theorem for (LF)-spaces

Let $E = \mathrm{ind}_{n\rightarrow} E_n$ be a boundedly retractive (LF)-space. Every bounded sequence in E has a subsequence which is $\sigma(E, E')$-Cauchy or equivalent to the unit vector basis of ℓ_1. In particular, E does not contain a copy of ℓ_1 if and only if every bounded sequence in E has a $\sigma(E, E')$-Cauchy subsequence.

Rosenthal's ℓ_1 theorem for (LF)-spaces (Proof)

- Let $\{x_j\}_j$ be a bounded sequence in E and assume that it has no $\sigma(E, E')$-Cauchy subsequence. There is $n_0 \in \mathbb{N}$ such that $\{x_j\}_j$ is a bounded sequence in E_{n_0}.

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- Now select $m \geq n_0$ such that E_m and E induce the same topology on the bounded sets of E_{n_0}.

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- Let $\{x_j\}_j$ be a bounded sequence in E and assume that it has no $\sigma(E, E')$-Cauchy subsequence. There is $n_0 \in \mathbb{N}$ such that $\{x_j\}_j$ is a bounded sequence in E_{n_0}.
- Now select $m \geq n_0$ such that E_m and E induce the same topology on the bounded sets of E_{n_0}.
- Since $\{x_j\}_j$ is bounded in E_m and it has no $\sigma(E_m, E'_m)$-Cauchy subsequence, we can apply Rosenthal's ℓ_1 theorem in the Fréchet space E_m to conclude that there is a subsequence $\{x_{j_k}\}_k$ which is equivalent to the unit vector basis of ℓ_1. That is, there exist c_1 and a continuous seminorm p in E_m such that

$$c_1 \sum_{k=1}^{\infty} |\alpha_k| \leq p\left(\sum_{k=1}^{\infty} \alpha_k x_{j_k}\right) \leq \sup_k p(x_{j_k}) \sum_{k=1}^{\infty} |\alpha_k|,$$

for every $\alpha = (\alpha_k)_k \in \ell_1$.

Rosenthal's ℓ_1 theorem for (LF)-spaces (Proof)

- Set $F := \{\sum_{k=1}^{\infty} \alpha_k x_{j_k} : \alpha = \{\alpha_k\}_k \in \ell_1\} \subset E_{n_0}$ and F endowed with any of them is a Banach space isomorphic to ℓ_1. The spaces E_{n_0} and E_m induce on F the same (Banach) topology.

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- Denote by U_F the closed unit ball of F and by τ_m and τ the topologies of E_m and E, respectively. Then τ and τ_m coincide on U_F, which is an absolutely convex 0-neighbourhood for $\tau_m|_F$.

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- Applying a result of Roelcke we conclude that τ_m and τ coincide in F; hence, there is a continuous seminorm r on E such that $p(z) \leq r(z)$ for every $z \in F$. This implies, for each $\alpha = (\alpha_k)_k \in \ell_1$,

$$c_1 \sum_{k=1}^{\infty} |\alpha_k| \leq p\left(\sum_{k=1}^{\infty} \alpha_k x_{j_k}\right) \leq r\left(\sum_{k=1}^{\infty} \alpha_k x_{j_k}\right) \leq \left(\sup_k r(x_{j_k})\right) \sum_{k=1}^{\infty} |\alpha_k| .$$

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- Set $F := \{\sum_{k=1}^{\infty} \alpha_k x_{j_k} : \alpha = \{\alpha_k\}_k \in \ell_1\} \subset E_{n_0}$ and F endowed with any of them is a Banach space isomorphic to ℓ_1. The spaces E_{n_0} and E_m induce on F the same (Banach) topology.
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- Thus, $\{x_{j_k}\}_k$ is equivalent to the unit vectors of ℓ_1 in E and the inclusion $F \hookrightarrow E$ is a topological isomorphism into. Then, E contains an isomorphic copy of ℓ_1.

James' type theorems

Theorem

Let E be a boundedly retractive (LF)-space. Assume that E admits an unconditional Schauder frame $(\{x_j'\}, \{x_j\})$. Then, $(\{x_j'\}, \{x_j\})$ is shrinking if and only if E does not contain a copy of ℓ_1.

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- If $(\{x_j'\}, \{x_j\})$ is shrinking, E_β' is separable.

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- If $(\{x_j'\}, \{x_j\})$ is shrinking, E_β' is separable. Therefore E contains no subspace isomorphic to ℓ_1.

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- If $(\{x_j'\}, \{x_j\})$ is shrinking, E'_β is separable. Therefore E contains no subspace isomorphic to ℓ_1.
- We know that $(\{x_j\}, \{x_j'\})$ is a Schauder frame of $(E', \sigma(E', E))$ and we prove that it is unconditional.

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- If $(\{x_j'\}, \{x_j\})$ is shrinking, E'_β is separable. Therefore E contains no subspace isomorphic to ℓ_1.
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- If $(\{x_j'\}, \{x_j\})$ is shrinking, E'_β is separable. Therefore E contains no subspace isomorphic to ℓ_1.
- We know that $(\{x_j\}, \{x_j'\})$ is a Schauder frame of $(E', \sigma(E', E))$ and we prove that it is unconditional. By Orlicz-Pettis' Theorem it is $\mu(E', E)$-unconditionally convergent to x'. By a result of *Bonet, Lindström (1993)*, if E does not contain a copy of ℓ_1 then every $\mu(E', E)$-null sequence in E' is strongly convergent to zero.

James' type theorems

Theorem

If E admits an unconditional Schauder frame $(\{x_j'\}, \{x_j\})$, then, $(\{x_j'\}, \{x_j\})$ is boundedly complete if and only if E does not contain a copy of c_0.

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Theorem

If E admits an unconditional Schauder frame $(\{x_j'\},\{x_j\})$, then, $(\{x_j'\},\{x_j\})$ is boundedly complete if and only if E does not contain a copy of c_0.

- If we suppose that E contains a copy of c_0, there exists a projection P such that $P(E) \simeq c_0$.

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- If we suppose that E contains a copy of c_0, there exists a projection P such that $P(E) \simeq c_0$. Then, c_0 is complemented in its bidual l_∞, a contradiction.
- Suppose that $(\{x'_j\}, \{x_j\})$ is not boundedly complete, then there exists U, a 0-neighborhood, and a sequence $\{y_j\}$ such that $p_U(y_j) \geq 1$ and $(y_j)_j$ converges to 0 in the topology $\sigma(E, E')$, a contradiction.

A Schauder frame for $C^\infty(K)$

Let K be a compact set in $\mathbb{R}^p$, $p \geq 1$, with $\mathring{K} \neq \emptyset$ such that $K = \overline{\mathring{K}}$.

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Let $C^\infty(K) := \{f \in C^\infty(\overset{\circ}{K}) : f$ and all its partial derivatives admit continuous extension to $K\}$.
The Fréchet space topology in $C^\infty(K)$ is defined by the seminorms:

$$q_n(f) := \sup\left\{\left|f^{(\alpha)}(x)\right| : x \in K,\ |\alpha| \leq n\right\}, n \in \mathbb{N}_0.$$

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Remark

No system of exponentials can be a basis in $C^\infty([0,1])$.

A Schauder frame for $C^\infty(K)$

Theorem

There exists a continuous linear extension operator $T : C^\infty(K) \to C^\infty(\mathbb{R}^p)$ if and only if there are sequences $(\lambda^j) \subset \mathbb{R}^p$ and $(u_j) \in C^\infty(K)'$ such that $(\{u_j\}, \{e^{2\pi i x\cdot\lambda^j}\})$ is a Schauder frame for $C^\infty(K)$.

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If we suppose that there exists a continuous linear extension operator.

- Let $M > 0$ such that $K \subset [-M, M]^p$.

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If we suppose that there exists a continuous linear extension operator.

- Let $M > 0$ such that $K \subset [-M, M]^p$.
- Choosing $\phi \in \mathcal{D}([-2M, 2M]^p)$ such that $\phi \equiv 1$ on a neighborhood of $[-M, M]^p$.

A Schauder frame for $C^\infty(K)$

Theorem

There exists a continuous linear extension operator $T : C^\infty(K) \to C^\infty(\mathbb{R}^p)$ if and only if there are sequences $(\lambda^j) \subset \mathbb{R}^p$ and $(u_j) \in C^\infty(K)'$ such that $(\{u_j\}, \{e^{2\pi i x\cdot\lambda^j}\})$ is a Schauder frame for $C^\infty(K)$.

If we suppose that there exists a continuous linear extension operator.

- Let $M > 0$ such that $K \subset [-M, M]^p$.
- Choosing $\phi \in \mathcal{D}([-2M, 2M]^p)$ such that $\phi \equiv 1$ on a neighborhood of $[-M, M]^p$.
- Let $f \in C^\infty(K)$ we define $Hf = \phi(T(f)) \in \mathcal{D}(]-2M, 2M[^p)$.

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- We extend Hf as a periodic C^∞ function in $\mathbb{R}^p$ and then we take $u_j(f) = a_j(Hf)$.

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To show the converse

- For every $f \in C^\infty(K)$ we have $f(x) = \sum_{j=1}^\infty u_j(f) e^{2\pi i x\cdot\lambda_j}$ in $C^\infty(K)$ and $\sum_{j=1}^\infty u_j(f) b_j e^{2\pi i x\cdot\lambda_j}$ converges in $C^\infty(K)$ for every $\{b_j\} \in \ell_\infty$.

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- After differentiation, we obtain that $\sum_{j=1}^{\infty} u_j(f) 2\pi b_j \lambda_j^\alpha e^{2\pi i x\cdot\lambda_j}$ converges in $C^\infty(K)$ for every $\alpha \in \mathbb{N}_0^p$ and $\{b_j\} \in \ell_\infty$.

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- In particular, this series converges for a fixed x_0 in the interior of K, from where it follows $\sum_{j=1}^\infty \left| u_j(f) 2\pi \lambda_j^\alpha \right| < +\infty$ for every $\alpha \in \mathbb{N}_0^p$.

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- The continuity of T follows from the Banach-Steinhaus theorem.

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The Schauder frame of $C^\infty(K)$ is shrinking and boundedly complete since $C^\infty(K)$ is a Montel space.

A Schauder frame for $C^\infty(K)$

If we assume that there exists a continuous linear extension operator, then, for a fixed $j_0 \in \mathbb{Z}^p$ we can choose ϕ such that the j_0-th Fourier coefficient of $\phi T(e^{2\pi i \lambda^{j_0} \cdot})$ is not equal to 1.

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Choosing $\psi \neq \phi$ in the proof above, we find a different sequence $(v_j) \in C^\infty(K)'$ such that $\left(\{v_j\}, \{e^{2\pi i x \cdot \lambda_j}\}\right)$ is an unconditional Schauder frame for $C^\infty(K)$.

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Definition

Let E be a lcs and Λ be a sequence space.

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Let Λ be a sequence lcs, a sequence $\{g_i\}_i \subset E'$ is called a Λ-*Bessel sequence* in E' if $U_{\{g_i\}_i} : E \to \Lambda$ defined by $U_{\{g_i\}_i}(x) := \{g_i(x)\}_i$ is a continuous linear operator.

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Definition

1. $\{g_i\}_i \subset E'$ is called a $\Lambda-$*frame* if the analysis operator is an isomorphism into the sequence space Λ.
2. $\{g_i\}_i \subset E'$ is called a *frame with respect to* Λ if the range of the analysis operator, $R\left(U_{\{g_i\}_i}\right)$, is complemented in Λ. This is, $U_{\{g_i\}_i}$ is a continuous operator and there exists $S : \Lambda \to E$ such that $S \circ U = id|_E$.

Relation between frames and Schauder frames

Let $(\{x'_n\}, \{x_n\})$ be a Schauder frame for a lcs E. Also, letting $\Lambda := \{\alpha = \{\alpha_j\}_j \in \omega : \sum_{j=1}^{\infty} \alpha_j x_j$ is convergent in $E\}$, then we obtain that $\{x'_i\}_i \subset E'$ is a $\Lambda-$frame and, indeed, is a frame with respect to Λ.

Relation between frames and Schauder frames

Let $(\{x_n'\}, \{x_n\})$ be a Schauder frame for a lcs E. Also, letting $\Lambda := \{\alpha = \{\alpha_j\}_j \in \omega : \sum_{j=1}^{\infty} \alpha_j x_j \text{ is convergent in } E\}$, then we obtain that $\{x_i'\}_i \subset E'$ is a $\Lambda-$frame and, indeed, is a frame with respect to Λ.

Moreover, if Λ has a Schauder basis $\{e_i\}_i$ and $\{g_i\}_i \subset E'$ is a frame with respect to Λ, then, there exists a Schauder frame for E.

Relation between frames and Schauder frames

Definition

We define β-dual space of a sequence space Λ as
$\Lambda^\beta := \{\{y_i\}_i \in \omega : \sum_{i=1}^\infty x_i y_i$ converges for every $\{x_i\}_i \in \Lambda\}$.

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Moreover, if $(\{x'_n\}, \{x_n\})$ is a Schauder frame for a barrelled lcs E and Λ is the associated sequence space given before then the operators $U : E \to \Lambda$ and $S : \Lambda \to E$ given by $U(x) := \{x'_n(x)\}_n$ and $S(\{\alpha_i\}_i) := \sum_{i=1}^{\infty} \alpha_i x_i$ respectively, are continuous operators such that $S \circ U = id_E$.

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If the canonical unit vectors $\{e_i\}_i$ form a Schauder basis and Λ is a barrelled sequence space then Λ'_β can be identified algebraically with the β-dual.

Relation between frames and Schauder frames

Also, $U' : \Lambda' \to E'$ and $S' : E' \to \Lambda'$ with $S'(x') := \{x'(x_i)\}_i$ are such that $U' \circ S' = id_{E'}$ and $S'(\Lambda')$ is complemented in Λ'. Indeed, given a bounded set $B \subset E$, $C := U(B)$ is bounded in Λ and $S'(E') \cap C^\circ \subset S'(B^\circ)$. Therefore, S' is an isomorphism into and, in many cases, the dual space has a frame with respect to Λ'.

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Note also that, if E'_β is not separable, then cannot have a Schauder frame.

Theorem

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Let E be a barrelled lcs and let Λ be a barrelled sequence lcs for which the canonical unit vectors $\{e_i\}_i$ form a Schauder basis. Then $\{g_i\}_i \subset E'$ is Λ'-Bessel for E if and only if the operator $T : \Lambda \to E'_\beta$ given by $T(\{d_i\}_i) := \sum_{i=1}^{\infty} d_i g_i$ is well defined and continuous.

Corollary

Let E be a barrelled lcs and let Λ be a barrelled locally convex sequence space, whose strong dual Λ'_β is barrelled and the canonical unit vectors $\{e_i\}_i$ form a Schauder basis. If $\{g_i\}_i \subset E'$ is Λ-Bessel for E then the operator $T : \Lambda' \to E'$ given by $T(\{d_i\}_i) := \sum_{i=1}^{\infty} d_i g_i$ is well defined and continuous. If Λ is reflexive, the converse holds.

Theorem

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Let E be a barrelled and complete lcs and let Λ be a barrelled sequence space for which the canonical unit vectors $\{e_i\}_i$ form a Schauder basis. If $\{g_i\}_i \subset E'$ is Λ-Bessel for E then the following conditions are equivalent:

(i) $\{g_i\}_i \subset E'$ is a frame with respect to Λ.

(ii) There exists a family $\{f_i\}_i \subset E$, such that $\sum_{i=1}^{\infty} c_i f_i$ is convergent for every $\{c_i\}_i \in \Lambda$ and $x = \sum_{i=1}^{\infty} g_i(x) f_i$, for every $x \in E$.

Example

Let $Exp(\mathbb{C})$ the vector space of entire functions of exponential type,

$$Exp(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) : \sup_{z\in\mathbb{C}} |f(z)|e^{-k|z|} < \infty \text{ for some } k > 0\}.$$

Let $\mathcal{K}$ denote the space of all positive continuous functions $h(z)$ on the complex plane $\mathbb{C}$ such that $\exp(A|z|) = O(h(z))$ as $|z| \to \infty$ for every $A > 0$. Then, we are going to use the following sequence space Λ defined by the inductive limit $\kappa_\infty(V) = ind_r \ell_\infty(v_r)$ where $(v_r(n,m))_{r\in\mathbb{N}}$ is a decreasing sequence of weights such that, if $s > r$, $\frac{v_s}{v_r} \in c_0(\mathbb{Z}^2)$:

$$\Lambda := \{\alpha := (\alpha_{n,m})_{n,m\in\mathbb{Z}} : \sup \frac{|\alpha_{n,m}|}{h(n+im)} < \infty \text{ for every } h \in \mathcal{K}\}.$$

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Using a result of Taylor.

Example

The evaluations $\{\delta_{n+im}\}_{n,m\in\mathbb{Z}}$ over the lattice points $\{n + im : n, m = 0, \pm1, \pm2, \dots\}$ is a Λ-frame for $Exp(\mathbb{C})$.

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Frames in Fréchet spaces

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