Ergodic theory and linear dynamics - Episode 2

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Summary of Episode 1

• We want to study, given $T \in \mathcal{L}(X)$, if there exists a measure $\mu$ on $X$ with full support such that $T$ is a measure-preserving and ergodic transformation with respect to $\mu$. 
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- $T \in \mathcal{L}(X)$ is measure-preserving if and only if $TRT^* = T$;
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• We will not characterize ergodicity, but two stronger notions.
Weakly mixing - Strongly mixing

$T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is \textbf{weakly mixing}

\[\iff \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mu(A \cap T^{-n}(B)) - \mu(A)\mu(B)| = 0 \quad (A, B \in \mathcal{B})\]

\[\iff \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left| \int_X f(T^n z) g(z) \, d\mu(z) - \int_X f \, d\mu \int_X g \, d\mu \right| = 0.\]

$T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ is \textbf{strongly mixing}

\[\iff \lim_{n \to \infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B) \quad (A, B \in \mathcal{B})\]

\[\iff \lim_{n \to \infty} \int_X f(T^n z) g(z) \, d\mu(z) = \int_X f \, d\mu \int_X g \, d\mu \quad (f, g \in L^2(X, \mu)).\]
The characterization of mixing properties

**Theorem (Rudnicki, 1993)**

Let $\mu$ be a Gaussian measure on $X$ with full support and covariance operator $R$. Let $T \in \mathcal{L}(X)$ be measure-preserving. The following are equivalent:

(i) $T$ is weakly mixing (strongly mixing, respectively) with respect to $\mu$;

(ii) For all $x^*, y^* \in X^*$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle RT^n(x^*), y^* \rangle| = 0$$

($\lim_{n \to \infty} \langle RT^n(x^*), y^* \rangle = 0$, respectively).
The difficult implication - I

Assumption:

\[ \lim_{n \to \infty} \langle RT^n(x^*), y^* \rangle = 0 \quad (x^*, y^*) \in X^*. \]

Conclusion:

\[ \lim_{n \to \infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B) \quad (A, B \in \mathcal{B}). \quad (1) \]
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\]

Definition

A \subset X is a **cylinder set** if there exist \( N \geq 1, (x_1^*, \ldots, x_N^*) \) a family of independent vectors of \( X^* \) and \( E \subset \mathbb{C}^N \) such that

\[
A = \{ x \in X; \ (\langle x_1^*, x \rangle, \ldots, \langle x_N^*, x \rangle) \in E \}.
\]
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It suffices to testify (1) for \( A, B \) cylinder sets.
The difficult implication - II

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Cylinder sets are "finite-dimensional sets"
The difficult implication - II

Definition

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$$A = \{ x \in X; (\langle x_1^*, x \rangle, \ldots, \langle x_N^*, x \rangle) \in E \}.$$

Cylinder sets are ”finite-dimensional sets” $\implies$ we need a finite-dimensional lemma.

Lemma

Let $(\nu_n)$ be a sequence of Gaussian measures on some finite-dimensional Banach space $E = \mathbb{C}^N$, and let $\nu$ be a Gaussian measure on $E$ with full support. Assume that $R_{\nu_n} \to R_\nu$ as $n \to \infty$. Then $\nu_n(Q) \to \nu(Q)$ for every Borel set $Q \subset E$. 
The difficult implication - III

Assumption:

$$\lim_{n \to \infty} \langle RT^n(x^*), y^* \rangle = 0 \quad (x^*, y^*) \in X^*.$$ 

$$A = \{ x \in X; \langle x_1^*, x \rangle, \ldots, \langle x_N^*, x \rangle \in E \}$$

$$B = \{ x \in X; \langle y_1^*, x \rangle, \ldots, \langle y_M^*, x \rangle \in F \}.$$ 

Aim:

$$\lim_{n \to \infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B).$$

Tool:

Lemma

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How to find an ergodic measure

Questions to solve

• What kind of measures shall we consider?
• Given $T \in \mathcal{L}(X)$ and a measure $\mu$ on $X$, how to prove that $T$ is a measure-preserving transformation?
• Given $T \in \mathcal{L}(X)$ and a measure $\mu$ on $X$, how to prove that $T$ is ergodic with respect to $\mu$?
• What kind of conditions on $T \in \mathcal{L}(X)$ ensures that we can construct a measure $\mu$ on $X$ such that the dynamical system $(T, \mu)$ is ergodic?
Having sufficiently many $\mathbb{T}$-eigenvectors

**Definition**
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A vector \( x \in X \) is a \( \mathbb{T} \)-\textit{eigenvector} for \( T \) if \( T(x) = \lambda x \) for some \( \lambda \in \mathbb{T} \).

Definition
A map \( E : \mathbb{T} \to X \) is a \textbf{perfectly spanning} \( \mathbb{T} \)-\textit{eigenvector field} provided
\begin{enumerate}
  \item \( E \in L^\infty(\mathbb{T}, X) \);
  \item \( \forall \lambda \in \mathbb{T}, \ TE(\lambda) = \lambda E(\lambda) \);
  \item For any \( A \subset \mathbb{T} \) with \( m(A) = 0 \), then \( \text{span}(E(\lambda); \ \lambda \in A) \) is dense in \( A \).
\end{enumerate}
How to find an ergodic measure

The operator $K_E$

Let $T \in \mathcal{L}(X)$ with a perfectly spanning $\mathbb{T}$-eigenvector field $E$. Define

$$K_E : L^2(\mathbb{T}, dm) \rightarrow X$$

$$f \mapsto \int_{\mathbb{T}} f(\lambda)E(\lambda)dm(\lambda)$$

and $R = K_EK_E^*$. Then

1. $K_E$ has dense range;
2. $TRT^* = T$;
3. For any $x^*, y^* \in X^*$, $\langle RT^{*n}(x^*), y^* \rangle \rightarrow 0$. 
The intertwining equation

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$$K_E : L^2(\mathbb{T}, dm) \rightarrow X$$

$$f \leftrightarrow \int_{\mathbb{T}} f(\lambda)E(\lambda)dm(\lambda)$$

$$V : L^2(\mathbb{T}, dm) \rightarrow L^2(\mathbb{T}, dm)$$

$$f \leftrightarrow zf$$
The intertwining equation

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\[ TK = KV. \]

\[ TK_E(f) = \int_{\mathbb{T}} f(\lambda) TE(\lambda) \, dm(\lambda) \]
\[ = \int_{\mathbb{T}} f(\lambda) \lambda E(\lambda) \, dm(\lambda) \]
\[ = KVf. \]
Comparison

Let $T \in \mathcal{L}(X)$ with a perfectly spanning $\mathbb{T}$-eigenvector field $E$, let $R = K_E K_E^*$ and let $\mu$ be a Gaussian measure on $X$. 

1. $\mu$ has full support if and only if $K \mu$ has dense range; 
2. $T$ is measure-preserving if and only if $TR = T R^* = T$; 
3. $T \in \mathcal{L}(X)$ is strongly-mixing with respect to $\mu$ if and only if $\lim_{n \to \infty} \langle R \mu T^n(x^*), y^* \rangle = 0$ ($x^*, y^*$) from $X$. 

Is $R$ the covariance operator of some Gaussian measure $\mu$?
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Is $R : X^* \to X$ a covariance operator?

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**Definition**

Let $\mathcal{H}$ be a separable Hilbert space. An operator $K \in \mathcal{L}(\mathcal{H}, X)$ is said to be \textbf{$\gamma$-radonifying} if for some (equivalently, for any) orthonormal basis $(e_n)$ of $\mathcal{H}$, the Gaussian series $\sum g_n(\omega)K(e_n)$ converges almost surely.

Proposition

Let $K \in \mathcal{L}(\mathcal{H}, X)$ be $\gamma$-radonifying. Then $R = KK^*$ is the covariance operator of some Gaussian measure $\mu$ on $X$. It suffices to take for $\mu$ the distribution of the Gaussian sum $\sum g_n(\omega)Ke_n$. 
How to find an ergodic measure

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On a Hilbert space

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**Fact.** On a Hilbert space, a Gaussian sum $\sum_n g_n x_n$ converges almost surely if and only if $\sum_n \|x_n\|^2 < +\infty$. 
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On a Hilbert space, \( \gamma \)-radonifying operators and Hilbert-Schmidt operators coincide!
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$$K_E : L^2(\mathbb{T}, dm) \rightarrow \mathcal{X}$$

$$f \mapsto \int_{\mathbb{T}} f(\lambda)E(\lambda)dm(\lambda)$$
How to find an ergodic measure

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It is Hilbert-Schmidt!

$R = K_E K_E^*$ is the covariance operator of some Gaussian measure!
Theorem on a Hilbert space

**Theorem (B. Grivaux (2006))**

Let $X$ be a separable Hilbert space and let $T \in \mathcal{L}(X)$ be such that $T$ has a perfectly spanning $\mathbb{T}$-eigenvectorfield. Then there exists a Gaussian measure $\mu$ on $X$ with full support, with respect to which $T$ is a strongly-mixing measure-preserving transformation.
What about Banach spaces?

$K \in \mathcal{L}(\mathcal{H}, X)$ is $\gamma$-radonifying if for some orthonormal basis $(e_n)$ of $\mathcal{H}$, the Gaussian series $\sum g_n(\omega)K(e_n)$ converges almost surely.
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**Definition**

A Banach space $X$ is said to have (Gaussian) type $p \in [1, 2]$ if

$$\left\| \sum_n g_n x_n \right\|_{L^2(\Omega, X)} \leq C \left( \sum_n \|x_n\|^p \right)^{\frac{1}{p}},$$

for some finite constant $C$ and every finite sequence $(x_n) \subset X$. 
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for some finite constant $C$ and every finite sequence $(x_n) \subset X$.

- A Hilbert space has type $2$;
- $L^p$-spaces have type $\min(p, 2)$;
- Any Banach space has type $1$;
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**Corollary**

Let $X$ be a Banach space with type $p$ and let $K \in \mathcal{L}(\mathcal{H}, X)$. Then $K$ is $\gamma$-radonifying as soon as $\sum_n \|Ke_n\|^p < +\infty$ for some orthonormal basis $(e_n)$ of $\mathcal{H}$. 
What about Banach spaces?

Theorem (B. Matheron, 2009)
Let $X$ be a separable Banach space and let $T \in \mathcal{L}(X)$ be such that $T$ has a perfectly spanning $\mathbb{T}$-eigenvector field $E$. Suppose moreover that:

- $X$ has type $p$;
- $E$ is $\alpha$-Hölderian for some $\alpha > \frac{1}{p} - \frac{1}{2}$.

Then there exists a Gaussian measure $\mu$ on $X$ with full support, with respect to which $T$ is a strongly-mixing measure-preserving transformation.
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One has to find an orthonormal basis $(e_n)$ of $L^2(\mathbb{T})$ such that $\sum_n \|K e_n\|^p < +\infty$. 

• $(e_n) = (e_{\text{int}})$ (B. Grivaux 2007). The result is less good.

• $(e_n) = \text{the Haar basis of } L^2(\mathbb{T})$. 

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Banach spaces=Hilbert spaces!

**Theorem (B. 2011)**

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**Strategy.** Instead of considering

$$K_E : L^2(\mathbb{T}, dm) \to X,$$

consider

$$K_E : L^2(\mathbb{T}, \sigma) \to X,$$

with $\sigma$ a continuous measure on $\mathbb{T}$. 

Theorem (B. 2011)
Let $X$ be a separable Banach space and let $T \in \mathcal{L}(X)$ be such that $T$ has a perfectly spanning $\mathbb{T}$-eigenvector field. Then there exists a Gaussian measure $\mu$ on $X$ with full support, with respect to which $T$ is a weakly-mixing measure-preserving transformation.

Strategy. Instead of considering

$$K_E : L^2(\mathbb{T}, dm) \rightarrow X,$$

consider

$$K_E : L^2(\mathbb{T}, \sigma) \rightarrow X,$$

with $\sigma$ a continuous measure on $\mathbb{T}$. $\sigma$ will be carried on Cantor set!
Cantor sets

Definition
A subset $\mathcal{C}$ of $\mathbb{T}$ is a **Cantor set** if it is the continuous image of $\{-1, 1\}^\mathbb{N}$.
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$\{-1, 1\}^\mathbb{N}$ will be endowed with its Haar measure $\nu$. $\nu$ is the tensor product $P_1 \otimes P_2 \otimes \ldots$, with, one each coordinate,

$$P_k(\{-1\}) = 1/2 \quad \text{and} \quad P_k(\{1\}) = 1/2.$$
An orthonormal basis of $L^2(\{-1, 1\}^\omega)$. 

Any $\omega \in \{-1, 1\}^\mathbb{N}$ can be written 

$$\omega = (\varepsilon_1(\omega), \varepsilon_2(\omega), \ldots).$$

Definition 

Let $A \subset \mathcal{P}_f(\mathbb{N})$. The **Walsh function** $w_A$ is defined by 

$$w_A(\omega) = \prod_{n \in A} \varepsilon_n(\omega).$$
An orthonormal basis of $L^2(\{-1, 1\}^\mathbb{N}, \nu)$.

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**Definition**

Let $A \subset \mathcal{P}_f(\mathbb{N})$. The **Walsh function** $w_A$ is defined by

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**Theorem**

$(w_A)_{A \in \mathcal{P}_f(\mathbb{N})}$ is an orthonormal basis of $L^2(\{-1, 1\}^\mathbb{N}, \nu)$. 
A new $\gamma$-radonifying operator

Lemma

Let $\phi : \{-1, 1\}^\mathbb{N} \to \mathcal{C}$ be an homeomorphism and let $\sigma$ be the image of the Haar measure $\nu$ on $\{-1, 1\}^\mathbb{N}$ by $\phi$. Let $u : \{-1, 1\}^\mathbb{N} \to X$ be a continuous function such that, for any $n \geq 1$, for any $(s_1, \ldots, s_{n-1}) \in \{-1, 1\}^{n-1}$, any $s', s'' \in \{-1, 1\}^\mathbb{N}$,

$$\|u(s_1, \ldots, s_{n-1}, 1, s') - u(s_1, \ldots, s_{n-1}, -1, s'')\| \leq 3^{-n}.$$ 

Let also $E = u \circ \phi^{-1}$. Then there exists an orthonormal basis $(e_n)$ of $L^2(\mathbb{T}, d\sigma)$ such that the operator $K_E : L^2(\mathbb{T}, d\sigma) \to X$ satisfies

$$\sum_n \|K_E(e_n)\| < +\infty.$$
What remains to be done

• Prove that, if $T$ admits a perfectly spanning $\mathbb{T}$-eigenvector field, then one can construct $\phi : \mathcal{C} \to \mathbb{T}$, $u : \mathcal{C} \to X$ such that

$$\|u(s_1, \ldots, s_{n-1}, 1, s') - u(s_1, \ldots, s_{n-1}, -1, s'')\| \leq 3^{-n}$$

and $u(s)$ is a $\mathbb{T}$-eigenvector with eigenvalue $\phi(s)$.
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• Prove that everything remains true with $K_E : L^2(\mathbb{T}, d\sigma) \rightarrow X$ instead of $K_E : L^2(\mathbb{T}, dm) \rightarrow X$. 
What remains to be done

• Prove that, if $T$ admits a perfectly spanning $\mathbb{T}$-eigenvector field, then one can construct $\phi : C \to \mathbb{T}$, $u : C \to X$ such that

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• Prove that everything remains true with $K_E : L^2(\mathbb{T}, d\sigma) \to X$ instead of $K_E : L^2(\mathbb{T}, dm) \to X$.

In fact, we will have to consider several such maps instead of one!


The construction of Cantor sets

Lemma

Let $T \in \mathcal{L}(X)$ with a perfectly spanning $\mathbb{T}$-eigenvector field. Let also $(\varepsilon_n)$ be a sequence of positive real numbers. There exist a sequence $(C_i)$ of subsets of $\mathbb{T}$, a sequence of homeomorphisms $(\phi_i)$ from $\{-1, 1\}^\mathbb{N}$ onto $C_i$ and a sequence of continuous functions $(u_i)$, $u_i : \{-1, 1\}^\mathbb{N} \to S_X$ such that, setting $E_i = u_i \circ \phi_i^{-1}$,

(a) for any $i \geq 1$ and any $\lambda \in C_i$, $TE_i(\lambda) = \lambda E_i(\lambda)$;

(b) $\text{span}(E_i(\lambda); \ i \geq 1, \ \lambda \in C_i)$ is dense in $X$;

(c) for any $n \geq 1$, any $(s_1, \ldots, s_{n-1}) \in \{-1, 1\}^{n-1}$, any $s', s'' \in \{-1, 1\}^\mathbb{N}$,

$$\|u_i(s_1, \ldots, s_{n-1}, 1, s') - u_i(s_1, \ldots, s_{n-1}, -1, s'')\| \leq \varepsilon_n.$$
Step 1 Since $E$ has a perfectly spanning $\mathbb{T}$-eigenvector field, there exists a sequence $(x_i)$ satisfying:

- each $x_i$ belongs to $S_X$, is a $\mathbb{T}$-eigenvector and the corresponding eigenvalues $(\lambda_i)$ are all different;
- each $x_i$ is a limit of a subsequence $(x_{n_k})_{k \geq 1}$;
- $\text{span}(x_i; \ i \geq 1)$ is dense in $X$. 

How to prove this?
How to find an ergodic measure

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Step 2 The construction...
Proof of the main result

We apply the previous lemma with $\varepsilon_n = 3^{-n}$. We get $C_i$, $u_i$, $E_i$, $\sigma_i$ and an orthonormal basis of $L^2(\mathbb{T}, d\sigma_i)$ such that

$$\sum_n \|K_{E_i}(e_{n,i})\| < +\infty.$$
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$$\sum_n \| K_{E_i}(e_{n,i}) \| < +\infty.$$ 

We set $\mathcal{H} = \bigoplus_{i \geq 1} L^2(\mathbb{T}, d\sigma_i)$ and let $K : \mathcal{H} \to X$ be defined by

$$K(\bigoplus_i f_i) = \sum_i \alpha_i K_{E_i}(f_i)$$ 

where $(\alpha_i)$ satisfies

(a) $\sum_i \alpha_i^2 \| E_i \|^2_{L^2(\mathbb{T}, \sigma_i, X)} < +\infty$
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(b) $\sum_i \alpha_i \sum_n \| K_{E_i}(e_{n,i}) \|_X < +\infty$, so that $K$ is $\gamma$-radonifying.
Proof of the main result

We apply the previous lemma with \( \varepsilon_n = 3^{-n} \). We get \( C_i, u_i, E_i, \sigma_i \) and an orthonormal basis of \( L^2(\mathbb{T}, d\sigma_i) \) such that

\[
\sum_{n} \| K_{E_i}(e_{n,i}) \| < +\infty.
\]

We set \( \mathcal{H} = \oplus_{i \geq 1} L^2(\mathbb{T}, d\sigma_i) \) and let \( K : \mathcal{H} \to X \) be defined by

\[
K(\oplus_i f_i) = \sum_i \alpha_i K_{E_i}(f_i)
\]

where \( (\alpha_i) \) satisfies

(a) \( \sum_i \alpha_i^2 \| E_i \|_{L^2(\mathbb{T}, \sigma_i, X)}^2 < +\infty \), so that \( K \) is well-defined;

(b) \( \sum_i \alpha_i \sum_n \| K_{E_i}(e_{n,i}) \|_X < +\infty \), so that \( K \) is \( \gamma \)-radonifying.

Everything works with \( R = KK^* \).
In fact, we have obtained the following statement:

**Theorem**

*Let* $T \in \mathcal{L}(X)$ *be such that, for any* $D \subset \mathbb{T}$ *countable,*

$$\ker(T - \lambda I; \lambda \in \mathbb{T}\setminus D)$$ *is a dense subset of* $X$. *Then there exists a Gaussian measure* $\mu$ *on* $X$ *with full support, with respect to which* $T$ *is a weakly-mixing measure-preserving transformation.*
Example - backward weighted shifts

Let $B_w$ be the **weighted backward shift** on $\ell^p(\mathbb{N})$ with weight sequence $(w_n)$:

$$B_w(x_0, x_1, \ldots) = (w_1 x_1, w_2 x_2, w_3 x_3, \ldots).$$
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Suppose that

$$\sum_{n \geq 1} \frac{1}{(w_1 \cdots w_n)^p} < \infty.$$
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There exists a Gaussian measure $\mu$ on $\ell^p(\mathbb{N})$ with full support, with respect to which $B_w$ is a measure-preserving and weakly mixing transformation.
The condition

\[ \sum_{n \geq 1} \frac{1}{(w_1 \cdots w_n)^p} < \infty. \]

ensures that \( B_w \) admit \( \mathbb{T} \)-eigenvectors:

\[ E(\lambda) := \sum_{n \geq 0} \frac{\lambda^n}{w_1 \cdots w_n} e_n. \]
How to find an ergodic measure

Example - backward weighted shifts

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ensures that $B_w$ admit $\mathbb{T}$-eigenvectors:

$$E(\lambda) := \sum_{n \geq 0} \frac{\lambda^n}{w_1 \cdots w_n} e_n.$$ 

This eigenvector field is perfectly spanning.
Example - backward weighted shifts

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is perfectly spanning. Pick \( y \in \ell^q \) such that

\[ \langle y, E(\lambda) \rangle = 0 \text{ a.e..} \]

Then

\[ g(\lambda) = \sum_n \frac{y_n}{w_1 \cdots w_n} \lambda^n = 0 \text{ a.e..} \]
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\[ \implies \hat{g}(n) = 0 \text{ for all } n \in \mathbb{N}. \]
Example - Adjoins of multipliers

\[ H^2(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C}; \|f\|^2_{H^2} := \sup_{r < 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \right\} \]

\[ = \left\{ f(z) = \sum_n a_n z^n; \sum_n |a_n|^2 < +\infty \right\}. \]

\[ H^\infty(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{C}; \|f\|_\infty < +\infty \}. \]

Definition
For \( \phi \in H^\infty(\mathbb{D}) \), the multiplier \( M_\phi \) is defined by \( M_\phi(f) = \phi f \), \( f \in H^2(\mathbb{D}) \).

Theorem
If \( \phi \) is non-constant and \( \phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset \), then there exists a Gaussian measure with full support on \( H^2(\mathbb{D}) \) with respect to which \( M_\phi^* \) is a measure-preserving and weakly mixing transformation.
Let $k_z$ be the reproducing kernel at $z \in \mathbb{D}$:

$$\forall f \in H^2(\mathbb{D}) : f(z) = \langle f, k_z \rangle_{H^2}.$$
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$k_z$ is an eigenvector for $M^*_\phi$.

$$\langle f, M^*_\phi(k_z) \rangle_{H^2} = \langle \phi f, k_z \rangle_{H^2} = \phi(z)f(z) = \langle f, \overline{\phi(z)}k_z \rangle_{H^2}.$$
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\]

When \( \phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset \), one can find an open arc \( I \subset \mathbb{T} \) and a curve \( \Gamma \subset \mathbb{D} \) such that \( \phi(\Gamma) = I \).
Adjointsofmultipliers

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When $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, one can find an open arc $I \subset \mathbb{T}$ and a curve $\Gamma \subset \mathbb{D}$ such that $\phi(\Gamma) = I$.

$$E(e^{i\theta}) := 1_I(e^{i\theta})k_{\phi^{-1}}(e^{i\theta}).$$  

is a (conjugate) $\mathbb{T}$-eigenvector field.
Adjoints of multipliers

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is perfectly spanning.
**Adjoint of multipliers**

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is perfectly spanning. Pick \( f \in H^2(\mathbb{D}) \) such that

\[ \langle f, E(e^{i\theta}) \rangle = 0 \text{ a.e.} \]
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\[ f \equiv 0. \]
And so on...

- Many other examples (composition operators,...);
- Many other results (about the converse, on semigroups of operators,...)
Muchas gracias!