Ergodic theory and linear dynamics - Episode 2

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February 2011

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$$\langle R(x^*), y^* \rangle = \int_X \overline{\langle x^*, z \rangle} \langle y^*, z \rangle \, d\mu(z);$$

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- μ has full support if and only if K has dense range;
- $T \in \mathfrak{L}(X)$ is measure-preserving if and only if $TRT^* = T$;
- We will not characterize ergodicity, but two stronger notions.

Weakly mixing - Strongly mixing

$T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is **weakly mixing**

$$\iff \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|\mu(A\cap T^{-n}(B))-\mu(A)\mu(B)|=0\ (A,B\in\mathcal{B})$$

$$\iff \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\left|\int_X f(T^n z)g(z)\,d\mu(z)-\int_X fd\mu\int_X gd\mu\right|=0.$$

$T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is **strongly mixing**

$$\iff \lim_{n\to\infty}\mu(A\cap T^{-n}(B))=\mu(A)\mu(B)\quad(A,B\in\mathcal{B})$$

$$\iff \lim_{n\to\infty}\int_X f(T^n z)g(z)\,d\mu(z)=\int_X f\,d\mu\int_X g\,d\mu\ (f,g\in L^2(X,\mu)).$$

The characterization of mixing properties

Theorem (Rudnicki, 1993)

Let μ be a Gaussian measure on X with full support and covariance operator R. Let $T \in \mathfrak{L}(X)$ be measure-preserving. The following are equivalent:

(i) *T is weakly mixing (strongly mixing, respectively) with respect to μ;*

(ii) *For all $x^*, y^* \in X^*$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle RT^{*n}(x^*), y^*\rangle| = 0$$

$\left(\lim_{n\to\infty} \langle RT^{*n}(x^*), y^*\rangle = 0\right.$, *respectively*$\left.\right)$.

The difficult implication - I

Assumption:

$$\lim_{n\to\infty} \langle RT^{*n}(x^*), y^* \rangle = 0 \ (x^*, y^*) \in X^*.$$

Conclusion:

$$\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B) \ (A, B \in \mathcal{B}). \tag{1}$$

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Definition

$A \subset X$ is a **cylinder set** if there exist $N \geq 1$, $(x_1^*, \dots, x_N^*)$ a family of independent vectors of X^* and $E \subset \mathbb{C}^N$ such that

$$A = \left\{x \in X;\ \left(\langle x_1^*, x \rangle, \dots, \langle x_N^*, x \rangle\right) \in E\right\}.$$

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It suffices to testify (1) for A, B cylinder sets.

The difficult implication - II

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Cylinder sets are "finite-dimensional sets"

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Cylinder sets are "finite-dimensional sets" $\implies$ we need a finite-dimensional lemma.

Lemma

Let (ν_n) be a sequence of Gaussian measures on some finite-dimensional Banach space $E = \mathbb{C}^N$, and let ν be a Gaussian measure on E with full support. Assume that $R_{\nu_n} \to R_\nu$ as $n \to \infty$. Then $\nu_n(Q) \to \nu(Q)$ for every Borel set $Q \subset E$.

The difficult implication - III

Assumption:

$$\lim_{n\to\infty} \langle RT^{*n}(x^*), y^* \rangle = 0 \ (x^*, y^*) \in X^*.$$

$$\begin{array}{rcl} A & = & \left\{x \in X;\ \left(\langle x_1^*, x\rangle, \dots, \langle x_N^*, x\rangle\right) \in E\right\} \\ B & = & \left\{x \in X;\ \left(\langle y_1^*, x\rangle, \dots, \langle y_M^*, x\rangle\right) \in F\right\}. \end{array}$$

Aim :

$$\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B).$$

Tool :

Lemma

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Questions to solve

- What kind of measures shall we consider?
- Given $T \in \mathfrak{L}(X)$ and a measure μ on X, how to prove that T is a measure-preserving transformation?
- Given $T \in \mathfrak{L}(X)$ and a measure μ on X, how to prove that T is ergodic with respect to μ?
- What kind of conditions on $T \in \mathfrak{L}(X)$ ensures that we can construct a measure μ on X such that the dynamical system (T, μ) is ergodic?

Having sufficiently many $\mathbb{T}$-eigenvectors

Definition

A vector $x \in X$ is a $\mathbb{T}$-**eigenvector** for T if $T(x) = \lambda x$ for some $\lambda \in \mathbb{T}$.

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Definition

A map $E : \mathbb{T} \to X$ is a **perfectly spanning** $\mathbb{T}$**-eigenvector field** provided

(i) $E \in L^\infty(\mathbb{T}, X)$;

(ii) $\forall \lambda \in \mathbb{T},\ \ TE(\lambda) = \lambda E(\lambda)$;

(iii) For any $A \subset \mathbb{T}$ with $m(A) = 0$, then $\mathrm{span}\big(E(\lambda);\ \lambda \in A\big)$ is dense in A.

The operator K_E

Let $T \in \mathfrak{L}(X)$ with a perfectly spanning $\mathbb{T}$-eigenvector field E. Define

$$\begin{aligned} K_E : L^2(\mathbb{T}, dm) &\to X \\ f &\mapsto \int_{\mathbb{T}} f(\lambda)E(\lambda)dm(\lambda) \end{aligned}$$

and $R = K_E K_E^*$. Then

1. K_E has dense range;
2. $TRT^* = T$;
3. For any $x^*, y^* \in X^*$, $\langle RT^{*n}(x^*), y^* \rangle \to 0$.

The intertwining equation

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$$\begin{aligned} TK_E(f) &= \int_{\mathbb{T}} f(\lambda)TE(\lambda)dm(\lambda) \\ &= \int_{\mathbb{T}} f(\lambda)\lambda E(\lambda)dm(\lambda) \\ &= KVf. \end{aligned}$$

Comparison

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   Ok for R!

Is R the covariance operator of some Gaussian measure μ?

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Definition

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Proposition

Let $K \in \mathfrak{L}(\mathcal{H}, X)$ be γ-radonifying. Then $R = KK^*$ is the covariance operator of some Gaussian measure μ on X.

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Proposition

Let $K \in \mathfrak{L}(\mathcal{H}, X)$ be γ-radonifying. Then $R = KK^*$ is the covariance operator of some Gaussian measure μ on X.

It suffices to take for μ the distribution of the Gaussian sum $\sum_n g_n K e_n$.

On a Hilbert space

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Fact. On a Hilbert space, a Gaussian sum $\sum_n g_n x_n$ converges almost surely if and only if $\sum_n \|x_n\|^2 < +\infty$.

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It is Hilbert-Schmidt!

$R = K_E K_E^*$ is the covariance operator of some Gaussian measure!

Theorem on a Hilbert space

Theorem (B. Grivaux (2006))

Let X be a separable Hilbert space and let $T \in \mathfrak{L}(X)$ be such that T has a perfectly spanning $\mathbb{T}$-eigenvectorfield. Then there exists a Gaussian measure μ on X with full support, with respect to which T is a strongly-mixing measure-preserving transformation.

What about Banach spaces?

$K \in \mathfrak{L}(\mathcal{H}, X)$ is γ-radonifying if for some orthonormal basis (e_n) of $\mathcal{H}$, the Gaussian series $\sum g_n(\omega)K(e_n)$ converges almost surely.

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Definition

A Banach space X is said to have (Gaussian) **type** $p \in [1, 2]$ if

$$\left\| \sum_n g_n x_n \right\|_{L^2(\Omega,X)} \leq C \left(\sum_n ||x_n||^p \right)^{\frac{1}{p}},$$

for some finite constant C and every finite sequence $(x_n) \subset X$.

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- A Hilbert space has type 2;
- L^p-spaces have type $\min(p, 2)$;
- Any Banach space has type 1;

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Corollary

Let X be a Banach space with type p and let $K \in \mathfrak{L}(\mathcal{H}, X)$. Then K is γ-radonifying as soon as $\sum_n \|Ke_n\|^p < +\infty$ for some orthonormal basis (e_n) of $\mathcal{H}$.

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Theorem (B. Matheron, 2009)

Let X be a separable Banach space and let $T \in \mathfrak{L}(X)$ be such that T has a perfectly spanning $\mathbb{T}$-eigenvector field E. Suppose moreover that :

- *X has type p;*
- *E is α-Hölderian for some $\alpha > \frac{1}{p} - \frac{1}{2}$.*

Then there exists a Gaussian measure μ on X with full support, with respect to which T is a strongly-mixing measure-preserving transformation.

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- $(e_n) = (e^{int})_{n\in\mathbb{Z}}$ (B. Grivaux 2007). The result is less good.
- $(e_n) =$ the Haar basis of $L^2(\mathbb{T})$.

Banach spaces=Hilbert spaces!

Theorem (B. 2011)

Let X be a separable Banach space and let $T \in \mathfrak{L}(X)$ be such that T has a perfectly spanning $\mathbb{T}$-eigenvector field. Then there exists a Gaussian measure μ on X with full support, with respect to which T is a weakly-mixing measure-preserving transformation.

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Strategy. Instead of considering

$$K_E : L^2(\mathbb{T}, dm) \to X,$$

consider

$$K_E : L^2(\mathbb{T}, \sigma) \to X,$$

with σ a continuous measure on $\mathbb{T}$.

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with σ a continuous measure on $\mathbb{T}$. σ will be carried on Cantor set!

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$\{-1, 1\}^{\mathbb{N}}$ will be endowed with its Haar measure ν. ν is the tensor product $\mathbb{P}_1 \otimes \mathbb{P}_2 \otimes \dots$, with, one each coordinate,

$$\mathbb{P}_k(\{-1\}) = 1/2 \text{ and } \mathbb{P}_k(\{1\}) = 1/2.$$

An orthonormal basis of $L^2(\{-1,1\}^\omega)$.

Any $\omega \in \{-1,1\}^{\mathbb{N}}$ can be written

$$\omega = (\varepsilon_1(\omega), \varepsilon_2(\omega), \dots).$$

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Let $A \subset \mathcal{P}_f(\mathbb{N})$. The **Walsh function** w_A is defined by

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$$w_A(\omega) = \prod_{n \in A} \varepsilon_n(\omega).$$

Theorem

$(w_A)_{A \in \mathcal{P}_f(\mathbb{N})}$ is an orthonormal basis of $L^2(\{-1,1\}^{\mathbb{N}}, \nu)$.

A new γ-radonifying operator

Lemma

Let $\phi : \{-1,1\}^{\mathbb{N}} \to \mathcal{C}$ be an homeomorphism and let σ be the image of the Haar measure ν on $\{-1,1\}^{\mathbb{N}}$ by ϕ. Let $u : \{-1,1\}^{\mathbb{N}} \to X$ be a continuous function such that, for any $n \geq 1$, for any $(s_1,\dots,s_{n-1}) \in \{-1,1\}^{n-1}$, any $s', s'' \in \{-1,1\}^{\mathbb{N}}$,

$$\|u(s_1,\dots,s_{n-1},1,s') - u(s_1,\dots,s_{n-1},-1,s'')\| \leq 3^{-n}.$$

Let also $E = u \circ \phi^{-1}$. Then there exists an orthonormal basis (e_n) of $L^2(\mathbb{T}, d\sigma)$ such that the operator $K_E : L^2(\mathbb{T}, d\sigma) \to X$ satisfies

$$\sum_n \|K_E(e_n)\| < +\infty.$$

What remains to be done

- Prove that, if T admits a perfectly spanning $\mathbb{T}$-eigenvector field, then one can construct $\phi : \mathcal{C} \to \mathbb{T}$, $u : \mathcal{C} \to X$ such that

$$\|u(s_1, \ldots, s_{n-1}, 1, s') - u(s_1, \ldots, s_{n-1}, -1, s'')\| \leq 3^{-n}$$

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In fact, we will have to consider several such maps instead of one!

The construction of Cantor sets

Lemma

Let $T \in \mathcal{L}(X)$ with a perfectly spanning $\mathbb{T}$-eigenvector field. Let also (ε_n) be a sequence of positive real numbers. There exist a sequence $(\mathcal{C}_i)$ of subsets of $\mathbb{T}$, a sequence of homeomorphisms (ϕ_i) from $\{-1,1\}^{\mathbb{N}}$ onto $\mathcal{C}_i$ and a sequence of continuous functions (u_i), $u_i : \{-1,1\}^{\mathbb{N}} \to S_X$ such that, setting $E_i = u_i \circ \phi_i^{-1}$,

(a) *for any $i \geq 1$ and any $\lambda \in \mathcal{C}_i$, $TE_i(\lambda) = \lambda E_i(\lambda)$;*

(b) *$span(E_i(\lambda);\ i \geq 1,\ \lambda \in \mathcal{C}_i)$ is dense in X;*

(c) *for any $n \geq 1$, any $(s_1, \dots, s_{n-1}) \in \{-1,1\}^{n-1}$, any $s', s'' \in \{-1,1\}^{\mathbb{N}}$,*

$$\|u_i(s_1, \dots, s_{n-1}, 1, s') - u_i(s_1, \dots, s_{n-1}, -1, s'')\| \leq \varepsilon_n.$$

How to prove this?

Step 1 Since E has a perfectly spanning $\mathbb{T}$-eigenvector field, there exists a sequence (x_i) satisfying :

- each x_i belongs to S_X, is a $\mathbb{T}$-eigenvector and the corresponding eigenvalues (λ_i) are all different;
- each x_i is a limit of a subsequence $(x_{n_k})_{k\geq 1}$;
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Step 2 The construction...

Proof of the main result

We apply the previous lemma with $\varepsilon_n = 3^{-n}$. We get $\mathcal{C}_i$, u_i, E_i, σ_i and an orthonormal basis of $L^2(\mathbb{T}, d\sigma_i)$ such that

$$\sum_n \|K_{E_i}(e_{n,i})\| < +\infty.$$

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We set $\mathcal{H} = \oplus_{i\geq 1} L^2(\mathbb{T}, d\sigma_i)$ and let $K : \mathcal{H} \to X$ be defined by

$$K(\oplus_i f_i) = \sum_i \alpha_i K_{E_i}(f_i)$$

where (α_i) satisfies

(a) $\sum_i \alpha_i^2 \|E_i\|^2_{L^2(\mathbb{T},\sigma_i,X)} < +\infty$

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Everything works with $R = KK^*$.

The correct statement

In fact, we have obtained the following statement:

Theorem

Let $T \in \mathfrak{L}(X)$ be such that, for any $D \subset \mathbb{T}$ countable, $\ker(T - \lambda I; \lambda \in \mathbb{T} \backslash D)$ is a dense subset of X. Then there exists a Gaussian measure μ on X with full support, with respect to which T is a weakly-mixing measure-preserving transformation.

Example - backward weighted shifts

Let $B_{\mathbf{w}}$ be the **weighted backward shift** on $\ell^p(\mathbb{N})$ with weight sequence (w_n):

$$B_{\mathbf{w}}(x_0, x_1, \ldots) = (w_1 x_1, w_2 x_2, w_3 x_3, \ldots).$$

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There exists a Gaussian measure μ on $\ell^p(\mathbb{N})$ with full support, with respect to which $B_{\mathbf{w}}$ is a measure-preserving and weakly mixing transformation.

Example - backward weighted shifts

The condition

$$\sum_{n\geq 1} \frac{1}{(w_1 \cdots w_n)^p} < \infty \, .$$

ensures that $B_{\mathbf{w}}$ admit $\mathbb{T}$-eigenvectors:

$$E(\lambda) := \sum_{n\geq 0} \frac{\lambda^n}{w_1 \cdots w_n} \, e_n \, .$$

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$$\langle y, E(\lambda)\rangle = 0 \text{ a.e..}$$

Then

$$g(\lambda) = \sum_n \frac{y_n}{w_1 \dots w_n} \lambda^n = 0 \text{ a.e..}$$

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$$\Longrightarrow \hat{g}(n) = 0 \text{ for all } n \in \mathbb{N}.$$

Example - Adjoints of multipliers

$$\begin{aligned} H^2(\mathbb{D}) &= \left\{ f : \mathbb{D} \to \mathbb{C};\, \|f\|_{H^2}^2 := \sup_{r<1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \right\} \\ &= \left\{ f(z) = \sum_n a_n z^n;\ \sum_n |a_n|^2 < +\infty \right\}. \end{aligned}$$

$$H^\infty(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{C};\ \|f\|_\infty < +\infty \}.$$

Definition

For $\phi \in H^\infty(\mathbb{D})$, the multiplier M_ϕ is defined by $M_\phi(f) = \phi f$, $f \in H^2(\mathbb{D})$.

Theorem

If ϕ is non-constant and $\phi(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$, then there exists a Gaussian measure with full support on $H^2(\mathbb{D})$ with respect to which M_ϕ^* is a measure-preserving and weakly mixing transformation.

Adjoints of multipliers

Let k_z be the **reproducing kernel** at $z \in \mathbb{D}$:

$$\forall f \in H^2(\mathbb{D}) \; : \; f(z) = \langle f, k_z \rangle_{H^2} \, .$$

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k_z is an eigenvector for M_ϕ^*.

$$\langle f, M_\phi^*(k_z) \rangle_{H^2} = \langle \phi f, k_z \rangle_{H^2} = \phi(z) f(z) = \langle f, \overline{\phi(z)} k_z \rangle_{H^2} .$$

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When $\phi(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$, one can find an open arc $I \subset \mathbb{T}$ and a curve $\Gamma \subset \mathbb{D}$ such that $\phi(\Gamma) = I$.

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$$E(e^{i\theta}) := \mathbf{1}_I(e^{i\theta}) k_{\phi^{-1}(e^{i\theta})}.$$

is a (conjugate) $\mathbb{T}$-eigenvector field.

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$$f \equiv 0.$$

And so on...

- Many other examples (composition operators,...);
- Many other results (about the converse, on semigroups of operators,...)

Muchas gracias!