Visualization of multi-objective optimization processes and asymmetric norms

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ABSTRACT

Asymmetric norms can be used in the mathematical development of specific tools for visualization of multi-objective optimization problems. The canonical asymmetric norm on a finite dimensional Banach lattice provides a topology that make compatible the arguments based on Euclidean distances and the notion of domination, that is often used in the study of Pareto sets in optimization problems. We show how this tool can be used for helping the decision maker to understand the information provided by a multi-objective optimization program.

- Multiobjective Optimization Problem.
- *The distance with respect to which the optimal solution is searched is not symmetric, in the sense that the distance from x to y does not coincide with the distance from y to x.*
- In this talk, we will show how to use this idea for an approximation of a set of solutions of the problem to a particular point with respect to a lattice asymmetric norm.
- Optimization argument based on the **domination** relation: a point x dominates other point y whenever y is $x + C$, where C is the positive cone of the lattice.

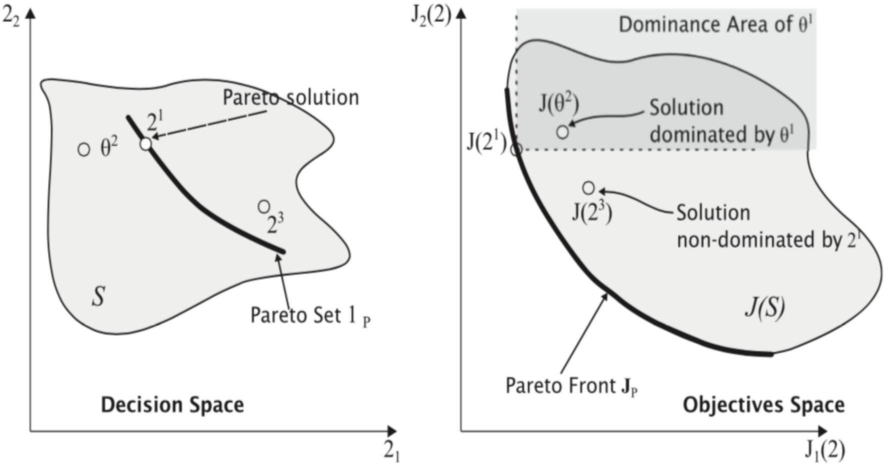

Fig. 1. Representation of Multiobjective problem concepts (Dominance, Pareto Front and Pareto Set) for 2D decision and objective spaces.

Figure : 1

We introduce a new technique for defining such an optimization structure, that has already shown to be useful for providing complementary information for visual representations of multivalued optimization problems.

From the formal point of view, the problem can be modeled as follows:

- We define a new order in the space given by the convex cone generated by the set D
- We define a quasi-distance d compatible with the ordering having the meaning that, if $x_0 \leq x$ $d(x_0, x)$ gives an Euclidean distance, and if $x \leq x_0$, then $d(x_0, x) = 0$.
- The optimization with respect to this new distance implies that there is a gain in minimizing the distance d from x_0 whenever the point $x \in A$ is as close as possible of satisfying $x \leq x_0$.

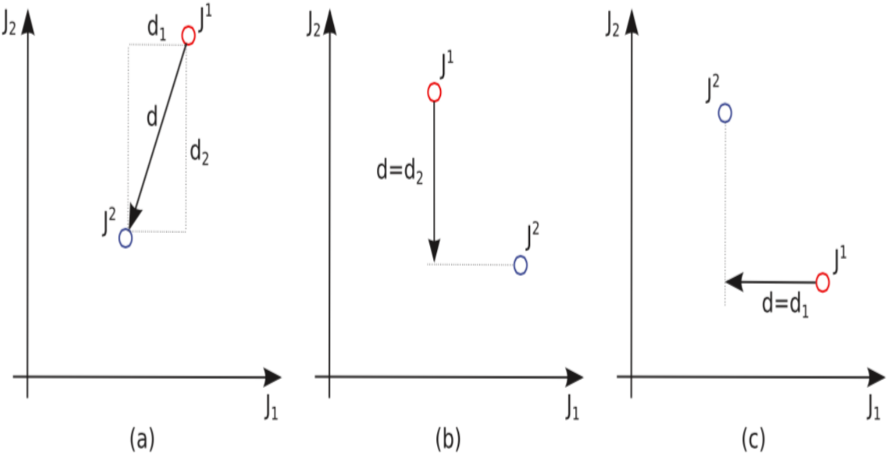

Fig. 5. $d(J^2, J^1)$ in a 2D space for different cases. How much is necessary to move J^1 to dominate J^2?

Figure : 2

FORMALIZATION

An *asymmetric norm* q is a real function $q : X \to [0, \infty)$ satisfying

1. $q(tx) = tq(x)$ for $t \geq 0$ and $x \in X$,
2. $q(x + y) \leq q(x) + q(y)$ and
3. $q(x) = 0 = q(-x)$ if and only if $x = -x = 0$.

A couple (X, q) is called an asymmetric normed linear space; it can be considered as a topological space with a non symmetric topology on X that is generated by the (non symmetric) open balls $B_q(x, \varepsilon) = \{y \in X \mid q(y - x) < \varepsilon\}$. Although this topology is not in general Hausdorff —in fact, this negative case is the one that is used in the present paper—, it always satisfies the separation axiom T_0.

FORMALIZATION

- There is a convex cone with vertex x such that any point of the cone is as good as x as a reference optimal point for getting the best approximation from A to the point x.
- The ordered induced by the convex cone in $\mathbb{R}^n$ models this situation: if a point $x \in A$ satisfies that $x \leq x_0$, then the solution provided by x is the better one. This order must be a lattice order in the finite dimensional space $\mathbb{R}^n$, and so for a pair of elements x, y in this space, the maximun $x \vee y$, the minimum $x \wedge y$ and the absolute value $|x|$ exist.
- We define a quasi-distance d given by an asymmetric norm q by $d(x, y) := q(y - x)$, that must be compatible with the order in the lattice sense, that is, if $x, y \in \mathbb{R}^n$ and $|x| \leq |y|$, then $q(x) \leq q(y)$. This norm must fit with the idea that for the optimization purposes, if $y \leq x$, then $d(x, y) = q(y - x) = 0$ that has been explained above.

Mathematical structure: asymmetric Euclidean lattices, that are finite dimensional spaces with an Euclidean norm $\|\cdot\|$ endowed with an ordering $\leq$, in such a way that the topology in the space is defined by the asymmetric norm given by

$$q(x) := \|x \vee 0\|, \quad x \in \mathbb{R}^n.$$

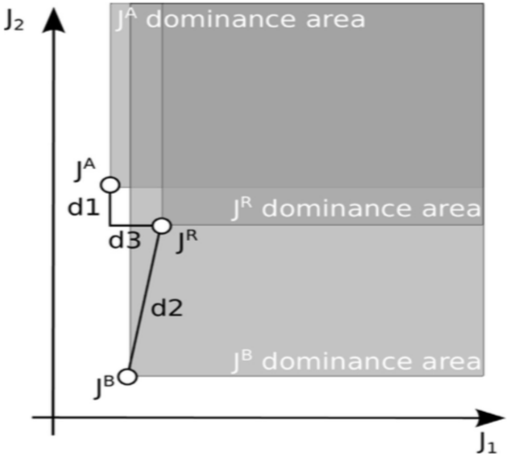

Fig. 7. An example where a point J^A is nearest to reference point J^R than J^B, but point J^B dominates J^R.

Figure : 3

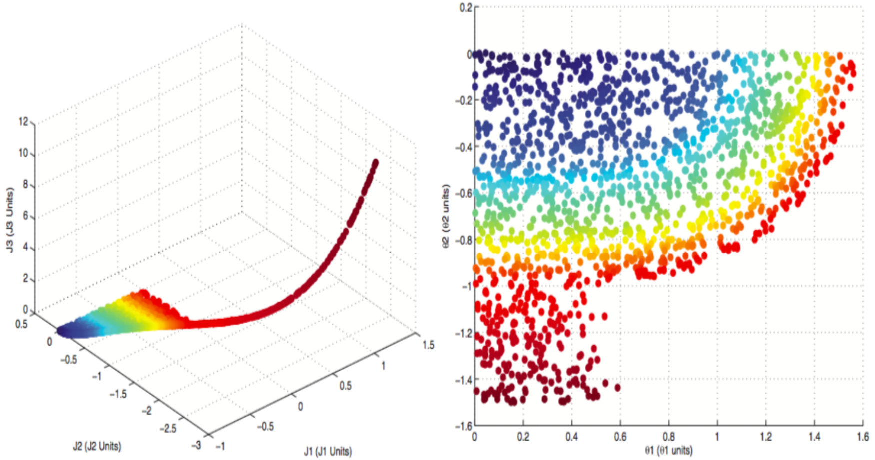

Fig. 2. 3D visualization for a Pareto front and 2D visualization for Pareto set. Three objectives ([$J1$, $J2$, $J3$]) and two decision parameters ([$\theta 1$, $\theta 2$]).

Figure : 4