Dominación y extensiones óptimas de operadores con rango esencial compacto en espacios de Banach de funciones

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Let $1 < p, p' < \infty$ such that $1/p + 1/p' = 1$.
We say that an operator T from $X(\mu)$ into a Banach space E satisfies a $1/p$-*th power approximation* if for each $\varepsilon > 0$ there is a function $h_\varepsilon \in X(\mu)_{[1/p']}$ such that for every function $f \in B_{X(\mu)}$ there is a function $g \in B_{X(\mu)_{[1/p]}}$ such that $\|T(f - gh_\varepsilon)\|_E < \varepsilon$.

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We say that an operator T from $X(\mu)$ into a Banach space E satisfies a *$1/p$-th power approximation* if for each $\varepsilon > 0$ there is a function $h_\varepsilon \in X(\mu)_{[1/p']}$ such that for every function $f \in B_{X(\mu)}$ there is a function $g \in B_{X(\mu)_{[1/p]}}$ such that $\|T(f - gh_\varepsilon)\|_E < \varepsilon$.

problem: *Let $X(\mu)$ be an order continuous Banach function space over a finite measure μ and let $T : X(\mu) \to E$ be a Banach space valued operator with compact essential range. When is the maximal extension of T compact?*

Motivation: Compactness. Grothendieck's result.

Theorem

Let E be a Banach space and let $K \subseteq E$ be a weakly closed subset. If for every $\varepsilon > 0$ there exists a weakly compact subset $K(\varepsilon)$ of X such that $K \subseteq K(\varepsilon) + \varepsilon B_E$, then K is weakly compact.

Theorem

Let $1 \leq p < \infty$ and X an order continuous Banach function space over a finite measure space. Let $T : X \to (E,\tau)$ be a continuous operator and let $\mathscr{U}$ be a basis of absolutely convex open neighborhoods of 0 in (E,τ). Consider the following assertions:

- **(a)** *The operator $T : X \to (E,\tau)$ is compact.*
- **(b)** *For each $g \in B_{X_{[1/p']}}$ the image $T(gB_{X_{[1/p]}})$ is relatively τ-compact in E and for every $U \in \mathscr{U}$ there exists $g_U \in X_{[1/p']}$ such that $T(B_X) \subset T(g_U B_{X_{[1/p]}}) + U$.*
- **(c)** *For each $g \in B_{X_{[1/p']}}$ the image $T(gB_{X_{[1/p]}})$ is relatively τ-compact in E and for every $U \in \mathscr{U}$ there exists $K_U > 0$ such that $T(B_X) \subset T(K_U B_{X_{[1/p]}}) + U$.*

Then (a) implies (b) and (c). If besides τ is metrizable then (b) or (c) implies (a).

Proof.

(a)⇒(b). If $T : X \to (E,\tau)$ is compact then $T(gB_{X_{[1/p]}})$ is relatively τ-compact in E for all $g \in B_{X_{[1/p']}}$.

Take $U \in \mathscr{U}$. Then $\overline{T(B_X)}^{\tau} \subset T(B_X) + U$

$$\subset T(\cup_{g \in B_{X_{[1/p']}}} gB_{X_{[1/p]}}) + U = \cup_{g \in B_{X_{[1/p']}}} (T(gB_{X_{[1/p]}}) + U).$$

By (a) $\overline{T(B_X)}^{\tau}$ is τ-compact. Since each set of the form $T(gB_{X_{[1/p]}}) + U$ is τ-open, there exist finitely many $g_1, \dots, g_l \in B_{X_{[1/p']}}$ such that

$$\overline{T(B_X)}^{\tau} \subset \cup_{i=1}^{l} T(g_i B_{X_{[1/p]}}) + U.$$

By the lattice properties of the norm of $X_{[1/p]}$, for any $h_1, h_2 \in X_{[1/p']}$ satisfying $|h_1| \le |h_2|$ a.e., we have that $h_1 B_{X_{[1/p]}} \subseteq |h_2| B_{X_{[1/p]}}$.

Define $g_U = \sum_{i=1}^{l} |g_i|$. Since $|g_i| \le g_U$, we obtain that $g_i B_{X_{[1/p]}} \subset g_U B_{X_{[1/p]}}$, and so $T(g_i B_{X_{[1/p]}}) \subset T(g_U B_{X_{[1/p]}})$ for all $i = 1, \dots, l$. Therefore,

$$\cup_{i=1}^{l} T(g_i B_{X_{[1/p]}}) + U \subset T(g_U B_{X_{[1/p]}}) + U$$

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(b)$\Rightarrow$(c) Let $\varepsilon > 0$ and $U \in \mathscr{U}$. Take $V \in \mathscr{U}$ such that $V + V \subset U$.

Consider $g_V \in X_{[1/p']}$ given by (b). By continuity there exists $\delta > 0$ such that $T(\delta B_X) \subset V$. Since simple functions are dense in $X_{[1/p']}$, we find a simple function s such that

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for every $f \in B_{X_{[1/p]}}$, and so $T((g_V - s)B_{X_{[1/p]}}) \subseteq V$.

Take $K_U := \|s\|_{L^\infty(\mu)}$. Then, again by the ideal property of $X_{[1/p]}$, we obtain

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Let us assume now that τ is metrizable. We prove (b)$\Rightarrow$(a) (the proof for (c)$\Rightarrow$(a) is the same).

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Lemma

Let $1 < p < \infty$. Let $X(\mu)$ be an order continuous Banach function space and E be a Banach space. A continuous operator $T : X(\mu) \to E$ is essentially compact if and only if for every $h \in X_{[1/p']}$ the map $T_h : X_{[1/p]} \to E$ given by $T_h(\cdot) := T(h\cdot)$, is compact.

Corollary

Let $1 < p < \infty$. Let $X(\mu)$ be an order continuous Banach function space and E be a Banach space. If $T : X(\mu) \to E$ is a continuous essentially compact operator then the restriction $T|_{X_{[1/p]}} : X_{[1/p]} \to E$ is compact.

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Theorem

Let $1 \leq p < \infty$ and let $X(\mu)$ be an order continuous Banach function space. The following statements for a continuous operator $T : X \to E$ are equivalent:

(i) *T is compact.*

(ii) *T is essentially compact and for every $\varepsilon > 0$ there exists $h_\varepsilon \in X_{[1/p']}$ such that $T(B_X) \subset T(h_\varepsilon B_{X_{[1/p]}}) + \varepsilon B_E$.*

(iii) *T is essentially compact and for every $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that $T(B_X) \subset T(K_\varepsilon B_{X_{[1/p]}}) + \varepsilon B_E$.*

Examples

(a) Take $X(\mu) = L^1[0,1]$ and $1 < p < \infty$. Then $X_{[1/p]} = L^p[0,1]$ isometrically. Consider a partition $\{A_i\}_{i=1}^{\infty}$ of $[0,1]$, where $\mu(A_i) > 0$ for all $i \in \mathbb{N}$. Take a sequence of non-null measurable sets $\{B_i\}_{i=1}^{\infty}$ such that $B_i \subseteq A_i$ for all i and $r_i := \mu(B_i)/\mu(A_i) \downarrow 0$ and the operator $T : L^1(\mu) \to \ell^1$ given by $T(f) := \sum_{i=1}^{\infty}(\int_{B_i} f\, d\mu) e_i \in \ell^1$, $f \in L^1[0,1]$. Let us show that the requirement on $\{T(\chi_A) : A \in \Sigma\}$ of being relatively compact is fulfilled. Consider the sequence $\{r_i e_i\}_{i=1}^{\infty}$. For every $A \in \Sigma$,

$$T(\chi_A) = \sum_{i=1}^{\infty} \mu(B_i \cap A) e_i \leq \sum_{i=1}^{\infty} \mu(A_i)(r_i e_i)$$

in the order of ℓ^1, and so each $T(\chi_A)$ is in the closure of the convex hull of $\{r_i e_i\}_{i=1}^{\infty}$. Thus, it is relatively compact. In fact, any ℓ^1-valued continuous operator defined on an order continuous Banach function space is essentially compact. However, since for every i we can find a norm one function f_i of $L^1(\mu)$ with support in B_i, the set $T(B_{L^1[0,1]})$ includes all $\{e_i\}_{i=1}^{\infty}$, and so T is not compact. Consequently, there is $\varepsilon > 0$ such that there is no K such that $T(B_{L^1[0,1]}) \subseteq K T(B_{L^p[0,1]}) + \varepsilon B_{\ell^p}$.

(b) Although operators of compact range with domain in an L^1-space allow sometimes good characterizations —due mainly to 1-concavity of L^1—, this fact is not crucial in the example above. Consider $p \geq q \geq 1$ and the class of spaces $E_{p,q} = (\oplus_p)_{i=1}^{\infty} L^q(A_i, \mu|_{A_i})$, where the $Ai's$ are chosen as in (a), with the norm

$$|||f|||_{p,q} := \Big(\sum_{i=1}^{\infty} \|f|_{A_i}\|^p_{L^q[0,1]} \Big)^{1/p}, \quad f \in E_{p,q}.$$

It is a Banach function space of Lebesgue measurable functions on $[0,1]$ over a measure μ_a given by $\mu_a(A) := \sum_{i=1}^{\infty} a_i \mu(A_i \cap A)$, where a is any strictly positive element of $B_{\ell^{p'}}$. Notice also that it contains $L^1[0,1]$. The same calculations that in the above case prove that T defined as in (a) is continuous also if defined from $E_{p,1}$ to ℓ^p, and its essential range T is relatively compact, but the operator is not compact. Then, since $(E_{p,1})_{[1/p]} = E_{p^2,p}$, we get that there is an $\varepsilon > 0$ such that $T(B_{E_{p,1}}) \nsubseteq KT(B_{E_{p^2,p}}) + \varepsilon B_{\ell^1}$ for all constants $K > 0$.

Corollary

Let $1 < p < \infty$. Let $X(\mu)$, E and T as above, and assume that T is essentially compact. Let $\Phi : B_X \to B_{X_{[1/p]}}$ be a function and suppose that

$$\lim_{K\to\infty} \sup_{f\in B_X} \left\| T(f\chi_{\{|f|\geq K|\Phi(f)|\}})\right\| = 0.$$

Then T is compact.

Corollary

Let T be an essentially compact (positive) kernel operator $T : X(\mu) \to Y(\nu)$ such that the kernel k satisfies that

$$\lim_{\mu(A)\to 0} \left\| \|\chi_A k(x,y)\|_{X'}(y)\right\|_Y = 0.$$

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Proposition. Let $1 \leq p < \infty$. The following are equivalent:

(a) The integration operator $I_m : L^1(m) \to (E, \| \ \|_E)$ is compact.

(b) $\mathscr{R}(m)$ is relatively compact and for every $\varepsilon > 0$ there exists $g_\varepsilon \in L^{p'}(m)$ such that $I_m(B_{L^1(m)}) \subset I_m(g_\varepsilon B_{L^p(m)}) + \varepsilon B_E$.

(c) $\mathscr{R}(m)$ is relatively compact and for every $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that $I_m(B_{L^1(m)}) \subset I_m(K_\varepsilon B_{L^p(m)}) + \varepsilon B_E$.

Let $1 < p, p' < \infty$ such that $1/p + 1/p' = 1$.
We say that an operator T from $X(\mu)$ into a Banach space E satisfies a *$1/p$-th power approximation* if for each $\varepsilon > 0$ there is a function $h_\varepsilon \in X(\mu)_{[1/p']}$ such that for every function $f \in B_{X(\mu)}$ there is a function $g \in B_{X(\mu)_{[1/p]}}$ such that $\|T(f - gh_\varepsilon)\|_E < \varepsilon$.

Corollary

Let $m : \Sigma \to E$ be a vector measure with relatively compact range. Suppose that the integration map $I_m : L^1(m) \to E$ satisfies an $1/p$-th power approximation. Then $L^1(m) = L^1(|m|)$.

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Theorem

Let $1 < p < \infty$, let $X(\mu)$ be an order continuous Banach function space, E a Banach space and $T : X(\mu) \to E$ a continuous operator. The following statements are equivalent for T.

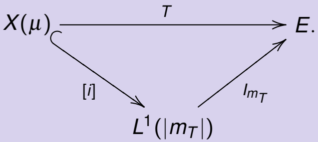
(i) *T is essentially compact, and for each $\varepsilon > 0$ there is $h_\varepsilon \in X_{[1/p']}$ such that*

$$\sup_{f \in B_{L^1(m_T)} \cap X(\mu)} \left(\inf_{g \in B_{L^p(m_T)} \cap X_{[1/p]}} \left(\left\| T(f - h_\varepsilon g) \right\|_E \right) \right) < \varepsilon.$$

(ii) *T is essentially compact, and for each $\varepsilon > 0$ there is a constant $K_\varepsilon > 0$ such that*

$$\sup_{f \in B_{L^1(m_T)} \cap X(\mu)} \left(\inf_{g \in B_{L^p(m_T)} \cap X_{[1/p]}} \left(\left\| T(f - K_\varepsilon g) \right\|_E \right) \right) < \varepsilon.$$

(iii) *The optimal domain of T is $L^1(|m_T|)$ and the extension I_{m_T} is compact, i.e. T factorizes compactly as*



Corollary

Let $1 < p < \infty$, let $X(\mu)$ be an order continuous Banach function space, E a Banach space and $T : X(\mu) \to E$ a μ-determined essentially compact continuous operator. The following statements are equivalent.

(i) *Each extension of T to an order continuous Banach function space satisfies a $1/p$-th power approximation.*

(ii) *I_{m_T} satisfies a $1/p$-th power approximation.*

(iii) *T admits a maximal extension that satisfies a $1/p$-th power approximation.*

We show a factorization theorem for homogeneous maps. The proof is based in a separation argument that has shown to be the key for proving factorization theorems for operators between Banach function spaces.

Proposition. Let E be a Banach space, and let $Y(\nu)$ and $Z(\nu)$ be Banach function spaces such that $Y(\nu) \subseteq Z(\nu)$. Let $U \subseteq E$ be an homogeneous set, and $\phi : U \to Y(\nu)$ and $P : U \to Z(\nu)$ be bounded homogeneous maps. Assume also that Y^Z has the Fatou property and $(Y^Z)'$ is order continuous. Then the following statements are equivalent.

(i) There is a constant $K > 0$ such that for every $x_1, \dots, x_n \in U$ and $A_1, \dots, A_n \in \Sigma$,

$$\| \sum_{i=1}^{n} |P(x_i)| \chi_{A_i} \|_{L^1(\nu)} \leq K \| \sum_{i=1}^{n} |\phi(x_i)| \chi_{A_i} \|_{(Y^Z)'}.$$

(ii) There is a function $g \in Y^Z$ such that for every $x \in U$ there is a function $h_x \in B_{L^\infty(\nu)}$ depending only on $x/\|x\|$ such that $P(x) = g \cdot h_x \cdot \phi(x)$. In other words, P factorizes through the homogeneous map $\hat{\phi}$ given by $\hat{\phi}(x) := h_x \cdot \phi(x)$ as

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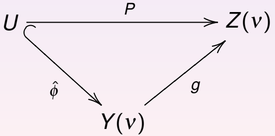

DEFINITIONS: Banach function spaces

- (Ω, Σ, μ) be a *finite* measure space.
- $L^0(\mu)$ space of all (classes of) measurable real functions on Ω.
- A *Banach function space* (briefly B.f.s.) is a Banach space $X \subset L^0(\mu)$ with norm $\|\cdot\|_X$ such that if $f \in L^0(\mu)$, $g \in X$ and $|f| \leq |g|$ μ-a.e. then $f \in X$ and $\|f\|_X \leq \|g\|_X$.
- A B.f.s. X has the *Fatou property* if for every sequence $(f_n) \subset X$ such that $0 \leq f_n \uparrow f$ μ-a.e. and $\sup_n \|f_n\|_X < \infty$, it follows that $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$.
- We will say that X is *order continuous* if for every $f, f_n \in X$ such that $0 \leq f_n \uparrow f$ μ-a.e., we have that $f_n \to f$ in norm.

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APPENDIX: DEFINITIONS. Vector measures and integration

- Let $m : \Sigma \to E$ be a *vector measure*, that is, a countably additive set function, where E is a real Banach space.

- A set $A \in \Sigma$ is *m-null* if $m(B) = 0$ for every $B \in \Sigma$ with $B \subset A$. For each x^* in the topological dual E^* of E, we denote by $|x^* m|$ the variation of the real measure $x^* m$ given by the composition of m with x^*. There exists $x_0^* \in E^*$ such that $|x_0^* m|$ has the same null sets as m. We will call $|x_0^* m|$ a *Rybakov control measure* for m.

- A measurable function $f : \Omega \to \mathbb{R}$ is *integrable with respect to m* if

(i) $\int |f|\, d|x^* m| < \infty$ for all $x^* \in E^*$.

(ii) For each $A \in \Sigma$, there exists $x_A \in E$ such that

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DEFINITIONS: Spaces of integrable functions

- Denote by $L^1(m)$ the space of integrable functions with respect to m, where functions which are equal m-a.e. are identified.
- The space $L^1(m)$ is a Banach space endowed with the norm

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Note that $L^\infty(|x_0^* m|) \subset L^1(m)$. In particular every measure of the type $|x^* m|$ is finite as $|x^* m|(\Omega) \leq \|x^*\| \cdot \|\chi_\Omega\|_m$.

- Given $f \in L^1(m)$, the set function $m_f : \Sigma \to E$ given by $m_f(A) = \int_A f \, dm$ for all $A \in \Sigma$ is a vector measure. Moreover, $g \in L^1(m_f)$ if and only if $gf \in L^1(m)$ and in this case $\int g \, dm_f = \int gf \, dm$.

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