Dominación y extensiones óptimas de operadores con rango esencial compacto en espacios de Banach de funciones

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Let $1 < p, p' < \infty$ such that $1/p + 1/p' = 1$.
We say that an operator $T$ from $X(\mu)$ into a Banach space $E$ satisfies a $1/p$-th power approximation if for each $\varepsilon > 0$ there is a function $h_\varepsilon \in X(\mu)[1/p]$ such that for every function $f \in B_{X(\mu)}$ there is a function $g \in B_{X(\mu)[1/p]}$ such that $\|T(f - gh_\varepsilon)\|_E < \varepsilon$.

**problem:** Let $X(\mu)$ be an order continuous Banach function space over a finite measure $\mu$ and let $T : X(\mu) \to E$ be a Banach space valued operator with compact essential range. When is the maximal extension of $T$ compact?
Let $1 < p, p' < \infty$ such that $1/p + 1/p' = 1$. We say that an operator $T$ from $X(\mu)$ into a Banach space $E$ satisfies a $1/p$-th power approximation if for each $\varepsilon > 0$ there is a function $h_{\epsilon} \in X(\mu)[1/p']$ such that for every function $f \in B_{X(\mu)}$ there is a function $g \in B_{X(\mu)[1/p]}$ such that $\|T(f - gh_{\epsilon})\|_E < \varepsilon$.

**Problem:** Let $X(\mu)$ be an order continuous Banach function space over a finite measure $\mu$ and let $T : X(\mu) \to E$ be a Banach space valued operator with compact essential range. When is the maximal extension of $T$ compact?
Motivation: Compactness. Grothendieck’s result.

**Theorem**

Let $E$ be a Banach space and let $K \subseteq E$ be a weakly closed subset. If for every $\varepsilon > 0$ there exists a weakly compact subset $K(\varepsilon)$ of $X$ such that $K \subseteq K(\varepsilon) + \varepsilon B_E$, then $K$ is weakly compact.
Theorem

Let $1 \leq p < \infty$ and $X$ an order continuous Banach function space over a finite measure space. Let $T : X \to (E, \tau)$ be a continuous operator and let $\mathcal{U}$ be a basis of absolutely convex open neighborhoods of 0 in $(E, \tau)$. Consider the following assertions:

(a) The operator $T : X \to (E, \tau)$ is compact.

(b) For each $g \in B_{X^{[1/p']}}$ the image $T(gB_{X^{[1/p']}})$ is relatively $\tau$-compact in $E$ and for every $U \in \mathcal{U}$ there exists $g_U \in X^{[1/p']}$ such that $T(B_X) \subset T(g_U B_{X^{[1/p']}}) + U$.

(c) For each $g \in B_{X^{[1/p']}}$ the image $T(gB_{X^{[1/p']}})$ is relatively $\tau$-compact in $E$ and for every $U \in \mathcal{U}$ there exists $K_U > 0$ such that $T(B_X) \subset T(K_U B_{X^{[1/p']}}) + U$.

Then (a) implies (b) and (c). If besides $\tau$ is metrizable then (b) or (c) implies (a).
Proof.

(a) ⇒ (b). If $T : X \to (E, \tau)$ is compact then $T(gB_{X_{[1/p]}})$ is relatively $\tau$-compact in $E$ for all $g \in B_{X_{[1/p']}}$.

Take $U \in \mathcal{U}$. Then $T(B_X)^\tau \subset T(B_X) + U \subset T(\bigcup_{g \in B_{X_{[1/p]}}} gB_{X_{[1/p]}}) + U = \bigcup_{g \in B_{X_{[1/p]}}}(T(gB_{X_{[1/p]}}) + U)$.

By (a) $T(B_X)^\tau$ is $\tau$-compact. Since each set of the form $T(gB_{X_{[1/p]}}) + U$ is $\tau$-open, there exist finitely many $g_1, \ldots, g_l \in B_{X_{[1/p]}}$ such that $T(B_X)^\tau \subset \bigcup_{i=1}^l T(g_iB_{X_{[1/p]}}) + U$.

By the lattice properties of the norm of $X_{[1/p]}$, for any $h_1, h_2 \in X_{[1/p]}$ satisfying $|h_1| \leq |h_2|$ a.e., we have that $h_1B_{X_{[1/p]}} \subseteq |h_2|B_{X_{[1/p]}}$.

Define $g_U = \sum_{i=1}^l |g_i|$. Since $|g_i| \leq g_U$, we obtain that $g_iB_{X_{[1/p]}} \subseteq g_UB_{X_{[1/p]}}$, and so $T(g_iB_{X_{[1/p]}}) \subset T(g_UB_{X_{[1/p]}})$ for all $i = 1, \ldots, l$. Therefore,

$$\bigcup_{i=1}^l T(g_iB_{X_{[1/p]}}) + U \subset T(g_UB_{X_{[1/p]}}) + U$$

and (b) is proved.
Proof.

(a)⇒(b). If \( T : X \rightarrow (E, \tau) \) is compact then \( T(gB_{X_{[1/p]}}) \) is relatively \( \tau \)-compact in \( E \) for all \( g \in B_{X_{[1/p]}} \).

Take \( U \in \mathcal{U} \). Then \( \overline{T(B_X)}^\tau \subset T(B_X) + U \)

\[
\subset T(\bigcup\{g \in B_{X_{[1/p]}} gB_{X_{[1/p]}}\}) + U = \bigcup\{g \in B_{X_{[1/p]}} T(gB_{X_{[1/p]}}) + U\}.
\]

By (a) \( \overline{T(B_X)}^\tau \) is \( \tau \)-compact. Since each set of the form \( T(gB_{X_{[1/p]}}) + U \) is \( \tau \)-open, there exist finitely many \( g_1, \ldots, g_l \in B_{X_{[1/p]}} \) such that

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\overline{T(B_X)}^\tau \subset \bigcup_{i=1}^l T(g_i B_{X_{[1/p]}}) + U.
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By the lattice properties of the norm of \( X_{[1/p]} \), for any \( h_1, h_2 \in X_{[1/p]} \) satisfying \( |h_1| \leq |h_2| \) a.e., we have that \( h_1 B_{X_{[1/p]}} \leq |h_2| B_{X_{[1/p]}} \).

Define \( g_U = \sum_{i=1}^l |g_i| \). Since \( |g_i| \leq g_U \), we obtain that \( g_i B_{X_{[1/p]}} \subset g_U B_{X_{[1/p]}} \), and so \( T(g_i B_{X_{[1/p]}}) \subset T(g_U B_{X_{[1/p]}}) \) for all \( i = 1, \ldots, l \). Therefore,

\[
\bigcup_{i=1}^l T(g_i B_{X_{[1/p]}}) + U \subset T(g_U B_{X_{[1/p]}}) + U
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Proof.

(a) ⇒ (b). If \( T : X \rightarrow (E, \tau) \) is compact then \( T(gB_{X[1/p]}) \) is relatively \( \tau \)-compact in \( E \) for all \( g \in B_{X[1/p']} \).

Take \( U \in \mathcal{U} \). Then \( \overline{T(B_X)^{\tau}} \subset T(B_X) + U \)

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By (a) \( \overline{T(B_X)^{\tau}} \) is \( \tau \)-compact. Since each set of the form \( T(gB_{X[1/p]}) + U \) is \( \tau \)-open, there exist finitely many \( g_1, \ldots, g_l \in B_{X[1/p']} \) such that

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\overline{T(B_X)^{\tau}} \subset \bigcup_{i=1}^{l} T(g_iB_{X[1/p]}) + U.
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By the lattice properties of the norm of \( X_{[1/p]} \), for any \( h_1, h_2 \in X_{[1/p']} \) satisfying \( |h_1| \leq |h_2| \) a.e., we have that \( h_1 B_{X[1/p]} \subseteq |h_2| B_{X[1/p]} \).

Define \( g_U = \sum_{i=1}^{l} |g_i| \). Since \( |g_i| \leq g_U \), we obtain that \( g_iB_{X[1/p]} \subseteq g_U B_{X[1/p]} \), and so \( T(g_iB_{X[1/p]}) \subseteq T(g_U B_{X[1/p]}) \) for all \( i = 1, \ldots, l \). Therefore,

\[
\bigcup_{i=1}^{l} T(g_iB_{X[1/p]}) + U \subset T(g_U B_{X[1/p]}) + U
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Proof.

(a) ⇒ (b). If \( T : X \rightarrow (E, \tau) \) is compact then \( T(gB_{X[1/p]}) \) is relatively \( \tau \)-compact in \( E \) for all \( g \in B_{X[1/p']} \).

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\bigcup_{i=1}^{l} T(g_iB_{X[1/p]}) + U \subset T(g_U B_{X[1/p]}) + U
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and (b) is proved.
Proof.

(a) ⇒ (b). If \( T : X \to (E, \tau) \) is compact then \( T(gB_{X[1/p]}) \) is relatively \( \tau \)-compact in \( E \) for all \( g \in B_{X[1/p']} \).

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\[
T(g_iB_{X[1/p]}) \subset T(g_U B_{X[1/p]})
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for all \( i = 1, \ldots, l \). Therefore,

\[
\bigcup_{i=1}^l T(g_iB_{X[1/p]}) + U \subset T(g_U B_{X[1/p]}) + U
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Proof.

(a)⇒(b). If $T : X \to (E, \tau)$ is compact then $T(gB_{X_{1/p}})$ is relatively $\tau$-compact in $E$ for all $g \in B_{X_{1/p'}}$.

Take $U \in \mathcal{U}$. Then $\overline{T(B_X)^\tau} \subset T(B_X) + U$

$$\subset T(\bigcup_{g \in B_{X_{1/p'}}} gB_{X_{1/p}}) + U = \bigcup_{g \in B_{X_{1/p'}}} (T(gB_{X_{1/p}}) + U).$$

By (a) $\overline{T(B_X)^\tau}$ is $\tau$-compact. Since each set of the form $T(gB_{X_{1/p}}) + U$ is $\tau$-open, there exist finitely many $g_1, \ldots, g_l \in B_{X_{1/p'}}$ such that

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By the lattice properties of the norm of $X_{1/p}$, for any $h_1, h_2 \in X_{1/p'}$ satisfying $|h_1| \leq |h_2|$ a.e., we have that $h_1B_{X_{1/p}} \subseteq |h_2|B_{X_{1/p}}$.

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$$\bigcup_{i=1}^l T(g_iB_{X_{1/p}}) + U \subset T(g_UB_{X_{1/p}}) + U,$$

and (b) is proved.
(b)⇒(c) Let $\varepsilon > 0$ and $U \in \mathcal{U}$. Take $V \in \mathcal{U}$ such that $V + V \subset U$.

Consider $g_V \in X_{1/p'}$ given by (b). By continuity there exists $\delta > 0$ such that $T(\delta B_X) \subset V$. Since simple functions are dense in $X_{1/p'}$, we find a simple function $s$ such that

$$
\|(g_V - s)f\|_X \leq \|g_V - s\|_{X_{1/p'}} < \delta
$$

for every $f \in B_{X_{1/p}}$, and so $T((g_V - s)B_{X_{1/p}}) \subset V$.

Take $K_U := \|s\|_{L^\infty(\mu)}$. Then, again by the ideal property of $X_{1/p}$, we obtain

$$
T(B_X) \subset T(g_VB_{X_{1/p}}) + V \subset T(sB_{X_{1/p}}) + T((g_V - s)B_{X_{1/p}}) + V
$$

$$
\subset T(sB_{X_{1/p}}) + V + V \subset T(K UB_{X_{1/p}}) + U. \text{ The proof is finished}
$$

Let us assume now that $\tau$ is metrizable. We prove (b)⇒(a) (the proof for (c)⇒(a) is the same).

Take an arbitrary $U \in \mathcal{U}$ and chose $g_U \in X_{1/p'}$ such that $T(B_X) \subset T(g_UB_{X_{1/p}}) + U$. By (b) the set $T(g_UB_{X_{1/p}})$ is relatively $\tau$-compact in $E$.

Then we have shown that for any $U \in \mathcal{U}$, there is a relatively $\tau$-compact set $T(g_UB_{X_{1/p}})$ such that $T(B_X) \subset T(g_UB_{X_{1/p}}) + U$. As $\tau$ is metrizable this implies that $T(B_X)$ is relatively $\tau$-compact.
(b)⇒(c) Let $\varepsilon > 0$ and $U \in \mathcal{U}$. Take $V \in \mathcal{U}$ such that $V + V \subset U$.

Consider $g_V \in X_{[1/p']}$ given by (b). By continuity there exists $\delta > 0$ such that $T(\delta B_X) \subset V$. Since simple functions are dense in $X_{[1/p']}$, we find a simple function $s$ such that

$$\| (g_V - s) f \|_X \leq \| g_V - s \|_{X_{[1/p']}} < \delta$$

for every $f \in B_{X_{[1/p]}}$, and so $T((g_V - s)B_{X_{[1/p]}}) \subseteq V$.

Take $K_U := \| s \|_{L^\infty(\mu)}$. Then, again by the ideal property of $X_{[1/p]}$, we obtain

$$T(B_X) \subseteq T(g_V B_{X_{[1/p]}}) + V \subseteq T(s B_{X_{[1/p]}}) + T((g_V - s) B_{X_{[1/p]}}) + V$$

$$\subseteq T(s B_{X_{[1/p]}}) + V + V \subseteq T(K_U B_{X_{[1/p]}}) + U.$$ The proof is finished.

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Take an arbitrary $U \in \mathcal{U}$ and chose $g_U \in X_{[1/p']}$ such that $T(B_X) \subset T(g_U B_{X_{[1/p]}}) + U$. By (b) the set $T(g_U B_{X_{[1/p]}})$ is relatively $\tau$-compact in $E$.

Then we have shown that for any $U \in \mathcal{U}$, there is a relatively $\tau$-compact set $T(g_U B_{X_{[1/p]}})$ such that $T(B_X) \subset T(g_U B_{X_{[1/p]}}) + U$. As $\tau$ is metrizable this implies that $T(B_X)$ is relatively $\tau$-compact.
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T(B_X) \subset T(g_V B_{X_{[1/p']}}) + V \subset T(s B_{X_{[1/p']}}) + T((g_V - s)B_{X_{[1/p']}}) + V
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$$
\subset T(s B_{X_{[1/p']}}) + V + V \subset T(K_U B_{X_{[1/p']}}) + U. \text{ The proof is finished}
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Let us assume now that $\tau$ is metrizable. We prove (b)⇒(a) (the proof for (c)⇒(a) is the same).

Take an arbitrary $U \in \mathcal{U}$ and chose $g_U \in X_{[1/p']}$ such that $T(B_X) \subset T(g_U B_{X_{[1/p']}}) + U$. By (b) the set $T(g_U B_{X_{[1/p']}})$ is relatively $\tau$-compact in $E$.

Then we have shown that for any $U \in \mathcal{U}$, there is a relatively $\tau$-compact set $T(g_U B_{X_{[1/p']}})$ such that $T(B_X) \subset T(g_U B_{X_{[1/p']}}) + U$. As $\tau$ is metrizable this implies that $T(B_X)$ is relatively $\tau$-compact.
(b) $\Rightarrow$ (c) Let $\varepsilon > 0$ and $U \in \mathcal{U}$. Take $V \in \mathcal{U}$ such that $V + V \subset U$.

Consider $g_V \in X_{[1/p']}^1$ given by (b). By continuity there exists $\delta > 0$ such that $T(\delta B_X) \subset V$. Since simple functions are dense in $X_{[1/p']}$, we find a simple function $s$ such that
\[
\|(g_V - s)f\|_X \leq \|g_V - s\|_{X_{[1/p']}} < \delta
\]
for every $f \in B_{X_{[1/p']}}$, and so $T((g_V - s)B_{X_{[1/p']}}) \subseteq V$.

Take $K_U := \|s\|_{L^\infty(\mu)}$. Then, again by the ideal property of $X_{[1/p]}$, we obtain
\[
T(B_X) \subseteq T(g_V B_{X_{[1/p']}}) + V \subseteq T(s B_{X_{[1/p']}}) + T((g_V - s)B_{X_{[1/p']}}) + V \\
\subseteq T(s B_{X_{[1/p']}}) + V + V \subseteq T(K_U B_{X_{[1/p']}}) + U. \text{ The proof is finished}
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Let us assume now that $\tau$ is metrizable. We prove (b) $\Rightarrow$ (a) (the proof for (c) $\Rightarrow$ (a) is the same).

Take an arbitrary $U \in \mathcal{U}$ and chose $g_U \in X_{[1/p']}$ such that $T(B_X) \subset T(g_U B_{X_{[1/p']}}) + U$. By (b) the set $T(g_U B_{X_{[1/p']}})$ is relatively $\tau$-compact in $E$.

Then we have shown that for any $U \in \mathcal{U}$, there is a relatively $\tau$-compact set $T(g_U B_{X_{[1/p']}})$ such that $T(B_X) \subset T(g_U B_{X_{[1/p']}}) + U$. As $\tau$ is metrizable this implies that $T(B_X)$ is relatively $\tau$-compact.
(b) $\Rightarrow$ (c) Let $\varepsilon > 0$ and $U \in \mathcal{U}$. Take $V \in \mathcal{U}$ such that $V + V \subset U$.

Consider $g_V \in X_{[1/p']}^*$ given by (b). By continuity there exists $\delta > 0$ such that $T(\delta B_X) \subset V$. Since simple functions are dense in $X_{[1/p']}$, we find a simple function $s$ such that

$$\|(g_V - s)f\|_X \leq \|g_V - s\|_{X_{[1/p']}} < \delta$$

for every $f \in B_{X_{[1/p]}}$, and so $T((g_V - s)B_{X_{[1/p]}}) \subset V$.

Take $K_U := \|s\|_{L^\infty(\mu)}$. Then, again by the ideal property of $X_{[1/p]}$, we obtain

$$T(B_X) \subset T(g_V B_{X_{[1/p]}}) + V \subset T(s B_{X_{[1/p]}}) + T((g_V - s) B_{X_{[1/p]}}) + V \subset T(s B_{X_{[1/p]}}) + V + V \subset T(K_U B_{X_{[1/p]}}) + U.$$ The proof is finished.

Let us assume now that $\tau$ is metrizable. We prove (b)$\Rightarrow$(a) (the proof for (c)$\Rightarrow$(a) is the same).

Take an arbitrary $U \in \mathcal{U}$ and chose $g_U \in X_{[1/p']}$ such that $T(B_X) \subset T(g_U B_{X_{[1/p]}}) + U$. By (b) the set $T(g_U B_{X_{[1/p]}})$ is relatively $\tau$-compact in $E$.

Then we have shown that for any $U \in \mathcal{U}$, there is a relatively $\tau$-compact set $T(g_U B_{X_{[1/p]}})$ such that $T(B_X) \subset T(g_U B_{X_{[1/p]}}) + U$. As $\tau$ is metrizable this implies that $T(B_X)$ is relatively $\tau$-compact.
(b)⇒(c) Let $\varepsilon > 0$ and $U \in \mathcal{U}$. Take $V \in \mathcal{U}$ such that $V + V \subseteq U$.

Consider $g_V \in X_{[1/p']}$ given by (b). By continuity there exists $\delta > 0$ such that $T(\delta B_X) \subseteq V$. Since simple functions are dense in $X_{[1/p']}$, we find a simple function $s$ such that

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for every $f \in B_{X_{[1/p']}}$, and so $T((g_V - s)B_{X_{[1/p']}}) \subseteq V$.

Take $K_U : = \| s \|_{L^\infty(\mu)}$. Then, again by the ideal property of $X_{[1/p]}$, we obtain

$$T(B_X) \subseteq T(g_V B_{X_{[1/p']}}) + V \subseteq T(sB_{X_{[1/p']}}) + T((g_V - s)B_{X_{[1/p']}}) + V$$

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Take an arbitrary $U \in \mathcal{U}$ and chose $g_U \in X_{[1/p']}$ such that $T(B_X) \subseteq T(g_U B_{X_{[1/p']}}) + U$. By (b) the set $T(g_U B_{X_{[1/p']}})$ is relatively $\tau$-compact in $E$.

Then we have shown that for any $U \in \mathcal{U}$, there is a relatively $\tau$-compact set $T(g_U B_{X_{[1/p']}})$ such that $T(B_X) \subseteq T(g_U B_{X_{[1/p']}}) + U$. As $\tau$ is metrizable this implies that $T(B_X)$ is relatively $\tau$-compact.
**Lemma**

Let $1 < p < \infty$. Let $X(\mu)$ be an order continuous Banach function space and $E$ be a Banach space. A continuous operator $T : X(\mu) \to E$ is essentially compact if and only if for every $h \in X_{[1/p']}$ the map $T_h : X_{[1/p]} \to E$ given by $T_h(\cdot) := T(h \cdot)$, is compact.

**Corollary**

Let $1 < p < \infty$. Let $X(\mu)$ be an order continuous Banach function space and $E$ be a Banach space. If $T : X(\mu) \to E$ is a continuous essentially compact operator then the restriction $T|_{X_{[1/p]}} : X_{[1/p]} \to E$ is compact.
Lemma

Let $1 < p < \infty$. Let $X(\mu)$ be an order continuous Banach function space and $E$ be a Banach space. A continuous operator $T : X(\mu) \to E$ is essentially compact if and only if for every $h \in X_{[1/p']}$ the map $T_h : X_{[1/p]} \to E$ given by $T_h(\cdot) := T(h \cdot)$, is compact.

Corollary

Let $1 < p < \infty$. Let $X(\mu)$ be an order continuous Banach function space and $E$ be a Banach space. If $T : X(\mu) \to E$ is a continuous essentially compact operator then the restriction $T|_{X_{[1/p]}} : X_{[1/p]} \to E$ is compact.
Theorem

Let \( 1 \leq p < \infty \) and let \( X(\mu) \) be an order continuous Banach function space. The following statements for a continuous operator \( T : X \to E \) are equivalent:

(i) \( T \) is compact.

(ii) \( T \) is essentially compact and for every \( \varepsilon > 0 \) there exists \( h_{\varepsilon} \in X_{1/p'} \) such that \( T(B_X) \subset T(h_{\varepsilon}B_{X_{1/p}}) + \varepsilon B_E \).

(iii) \( T \) is essentially compact and for every \( \varepsilon > 0 \) there exists \( K_{\varepsilon} > 0 \) such that \( T(B_X) \subset T(K_{\varepsilon}B_{X_{1/p}}) + \varepsilon B_E \).
(a) Take $X(\mu) = L^1[0, 1]$ and $1 < p < \infty$. Then $X_{[1/p]} = L^p[0, 1]$ isometrically. Consider a partition $\{A_i\}_{i=1}^\infty$ of $[0, 1]$, where $\mu(A_i) > 0$ for all $i \in \mathbb{N}$. Take a sequence of non-null measurable sets $\{B_i\}_{i=1}^\infty$ such that $B_i \subseteq A_i$ for all $i$ and $r_i := \mu(B_i)/\mu(A_i) \downarrow 0$ and the operator $T : L^1(\mu) \to \ell^1$ given by $T(f) := \sum_{i=1}^\infty (\int_{B_i} f \, d\mu) e_i \in \ell^1$, $f \in L^1[0, 1]$. Let us show that the requirement on $\{T(\chi_A) : A \in \Sigma\}$ of being relatively compact is fulfilled. Consider the sequence $\{r_i e_i\}_{i=1}^\infty$. For every $A \in \Sigma$,

$$T(\chi_A) = \sum_{i=1}^\infty \mu(B_i \cap A) e_i \leq \sum_{i=1}^\infty \mu(A_i)(r_i e_i)$$

in the order of $\ell^1$, and so each $T(\chi_A)$ is in the closure of the convex hull of $\{r_i e_i\}_{i=1}^\infty$. Thus, it is relatively compact. In fact, any $\ell^1$-valued continuous operator defined on an order continuous Banach function space is essentially compact. However, since for every $i$ we can find a norm one function $f_i$ of $L^1(\mu)$ with support in $B_i$, the set $T(B_{L^1}[0, 1])$ includes all $\{e_i\}_{i=1}^\infty$, and so $T$ is not compact. Consequently, there is $\epsilon > 0$ such that there is no $K$ such that $T(B_{L^1}[0, 1]) \subseteq KT(B_{L^p}[0, 1]) + \epsilon B_{\ell^p}$. 

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**Examples**

Theorem 1. Let $X(\mu) = L^1[0, 1]$ and $1 < p < \infty$. Then $X_{[1/p]} = L^p[0, 1]$ isometrically.
(b) Although operators of compact range with domain in an $L^1$-space allow sometimes good characterizations —due mainly to 1-concavity of $L^1$—, this fact is not crucial in the example above. Consider $p \geq q \geq 1$ and the class of spaces $E_{p,q} = (\oplus_p)_{i=1}^{\infty} L^q(A_i, \mu|_{A_i})$, where the $A_i$’s are chosen as in (a), with the norm

$$\|f\|_{p,q} := \left( \sum_{i=1}^{\infty} \|f|_{A_i}\|_L^p L^q(0,1) \right)^{1/p}, \quad f \in E_{p,q}.$$ 

It is a Banach function space of Lebesgue measurable functions on $[0,1]$ over a measure $\mu_a$ given by $\mu_a(A) := \sum_{i=1}^{\infty} a_i \mu(A_i \cap A)$, where $a$ is any strictly positive element of $B_{\ell^p}$. Notice also that it contains $L^1[0,1]$. The same calculations that in the above case prove that $T$ defined as in (a) is continuous also if defined from $E_{p,1}$ to $\ell^p$, and its essential range $T$ is relatively compact, but the operator is not compact. Then, since $(E_{p,1})_{[1/p]} = E_{p^2,p}$, we get that there is an $\varepsilon > 0$ such that $T(B_{E_{p,1}}) \notin KT(B_{E_{p^2,p}} + \varepsilon B_{\ell^1})$ for all constants $K > 0$. 

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Corollary

Let $1 < p < \infty$. Let $X(\mu)$, $E$ and $T$ as above, and assume that $T$ is essentially compact. Let $\Phi : B_X \to B_X[1/p]$ be a function and suppose that

$$\lim_{K \to \infty} \sup_{f \in B_X} \left\| T(f \chi_{\{ |f| \geq K \Phi(f) \}}) \right\| = 0.$$ 

Then $T$ is compact.

Corollary

Let $T$ be an essentially compact (positive) kernel operator $T : X(\mu) \to Y(\nu)$ such that the kernel $k$ satisfies that

$$\lim_{\mu(A) \to 0} \left\| \chi_A k(x, y) \chi'(y) \right\|_Y = 0.$$ 

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Then $T$ is compact.
Proposition. Let $1 \leq p < \infty$. The following are equivalent:

(a) The integration operator $I_m : L^1(m) \to (E, \| \cdot \|_E)$ is compact.

(b) $\mathcal{R}(m)$ is relatively compact and for every $\varepsilon > 0$ there exists $g_\varepsilon \in \mathcal{L}^p(m)$ such that

$$I_m(B_{L^1(m)}) \subset I_m(g_\varepsilon B_{L^p(m)}) + \varepsilon B_E.$$

(c) $\mathcal{R}(m)$ is relatively compact and for every $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that

$$I_m(B_{L^1(m)}) \subset I_m(K_\varepsilon B_{L^p(m)}) + \varepsilon B_E.$$
Let $1 < p, p' < \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$.
We say that an operator $T$ from $X(\mu)$ into a Banach space $E$ satisfies a $1/p$-th power approximation if for each $\varepsilon > 0$ there is a function $h_\varepsilon \in X(\mu)[1/p]$ such that for every function $f \in B_X(\mu)$ there is a function $g \in B_X(\mu)[1/p]$ such that $\|T(f - gh_\varepsilon)\|_E < \varepsilon$.

**Corollary**

Let $m : \Sigma \to E$ be a vector measure with relatively compact range. Suppose that the integration map $I_m : L^1(m) \to E$ satisfies an $1/p$-th power approximation. Then $L^1(m) = L^1(|m|)$.
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Corollary

Let $m : \Sigma \to E$ be a vector measure with relatively compact range. Suppose that the integration map $I_m : L^1(m) \to E$ satisfies an $1/p$-th power approximation. Then $L^1(m) = L^1(|m|)$. 
Theorem

Let $1 < p < \infty$, let $X(\mu)$ be an order continuous Banach function space, $E$ a Banach space and $T : X(\mu) \to E$ a continuous operator. The following statements are equivalent for $T$.

(i) $T$ is essentially compact, and for each $\varepsilon > 0$ there is $h_\varepsilon \in X[1/p']$ such that

$$\sup_{f \in B_{L^1(m_T)} \cap X(\mu)} \left( \inf_{g \in B_{L^p(m_T)} \cap X[1/p]} \left( \| T(f - h_\varepsilon g) \|_E \right) \right) < \varepsilon.$$

(ii) $T$ is essentially compact, and for each $\varepsilon > 0$ there is a constant $K_\varepsilon > 0$ such that

$$\sup_{f \in B_{L^1(m_T)} \cap X(\mu)} \left( \inf_{g \in B_{L^p(m_T)} \cap X[1/p]} \left( \| T(f - K_\varepsilon g) \|_E \right) \right) < \varepsilon.$$

(iii) The optimal domain of $T$ is $L^1(|m_T|)$ and the extension $l_{m_T}$ is compact, i.e. $T$ factorizes compactly as

$$X(\mu) \xrightarrow{i} L^1(|m_T|) \xrightarrow{l_{m_T}} E.$$
Corollary

Let $1 < p < \infty$, let $X(\mu)$ be an order continuous Banach function space, $E$ a Banach space and $T : X(\mu) \to E$ a $\mu$-determined essentially compact continuous operator. The following statements are equivalent.

(i) Each extension of $T$ to an order continuous Banach function space satisfies a $1/p$-th power approximation.

(ii) $I_{mT}$ satisfies a $1/p$-th power approximation.

(iii) $T$ admits a maximal extension that satisfies a $1/p$-th power approximation.
Extensiones óptimas de operadores
We show a factorization theorem for homogeneous maps. The proof is based in a separation argument that has shown to be the key for proving factorization theorems for operators between Banach function spaces.

**Proposition.** Let $E$ be a Banach space, and let $Y(\nu)$ and $Z(\nu)$ be Banach function spaces such that $Y(\nu) \subseteq Z(\nu)$. Let $U \subseteq E$ be an homogeneous set, and $\phi : U \rightarrow Y(\nu)$ and $P : U \rightarrow Z(\nu)$ be bounded homogeneous maps. Assume also that $YZ$ has the Fatou property and $(YZ)'$ is order continuous. Then the following statements are equivalent.

(i) There is a constant $K > 0$ such that for every $x_1, \ldots, x_n \in U$ and $A_1, \ldots, A_n \in \Sigma$,

$$\left\| \sum_{i=1}^{n} |P(x_i)| \chi_{A_i} \right\|_{L^1(\nu)} \leq K \left\| \sum_{i=1}^{n} |\phi(x_i)| \chi_{A_i} \right\|_{(YZ)'}.$$ 

(ii) There is a function $g \in YZ$ such that for every $x \in U$ there is a function $h_x \in B_{L^\infty(\nu)}$ depending only on $x/\|x\|$ such that $P(x) = g \cdot h_x \cdot \phi(x)$. In other words, $P$ factorizes through the homogeneous map $\hat{\phi}$ given by $\hat{\phi}(x) := h_x \cdot \phi(x)$ as:

$$
\begin{array}{ccc}
U & \xrightarrow{P} & Z(\nu) \\
\scriptstyle{\phi} \downarrow & & \downarrow \scriptstyle{g} \\
Y(\nu) & \scriptstyle{\hat{\phi}} \swarrow & \\
\end{array}
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$$U \xrightarrow{P} Z(\nu) \xrightarrow{\hat{\phi}} Y(\nu) \xleftarrow{\phi} g$$
DEFINITIONS: Banach function spaces

- \((\Omega, \Sigma, \mu)\) be a finite measure space.

- \(L^0(\mu)\) space of all (classes of) measurable real functions on \(\Omega\).

- A Banach function space (briefly B.f.s.) is a Banach space \(X \subset L^0(\mu)\) with norm \(\|\cdot\|_X\) such that if \(f \in L^0(\mu), g \in X\) and \(|f| \leq |g|\) \(\mu\)-a.e. then \(f \in X\) and \(\|f\|_X \leq \|g\|_X\).

- A B.f.s. \(X\) has the Fatou property if for every sequence \((f_n) \subset X\) such that \(0 \leq f_n \uparrow f\) \(\mu\)-a.e. and \(\sup_n \|f_n\|_X < \infty\), it follows that \(f \in X\) and \(\|f_n\|_X \uparrow \|f\|_X\).

- We will say that \(X\) is order continuous if for every \(f, f_n \in X\) such that \(0 \leq f_n \uparrow f\) \(\mu\)-a.e., we have that \(f_n \rightharpoonup f\) in norm.
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A set $A \in \Sigma$ is $m$-null if $m(B) = 0$ for every $B \in \Sigma$ with $B \subset A$. For each $x^*$ in the topological dual $E^*$ of $E$, we denote by $|x^* m|$ the variation of the real measure $x^* m$ given by the composition of $m$ with $x^*$. There exists $x_0^* \in E^*$ such that $|x_0^* m|$ has the same null sets as $m$. We will call $|x_0^* m|$ a Rybakov control measure for $m$.

A measurable function $f: \Omega \to \mathbb{R}$ is integrable with respect to $m$ if

(i) $\int |f| d|x^* m| < \infty$ for all $x^* \in E^*$.

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DEFINITIONS: Spaces of integrable functions

- Denote by $L^1(m)$ the space of integrable functions with respect to $m$, where functions which are equal $m$-a.e. are identified.

- The space $L^1(m)$ is a Banach space endowed with the norm
  \[ \|f\|_m = \sup_{x^* \in B_E^*} \int |f| \, d|x^*|m. \]

Note that $L^\infty(|x^*_0m|) \subset L^1(m)$. In particular every measure of the type $|x^*m|$ is finite as $|x^*m|(\Omega) \leq \|x^*\| \cdot \|\chi_\Omega\|_m$.

- Given $f \in L^1(m)$, the set function $m_f : \Sigma \rightarrow E$ given by $m_f(A) = \int_A f \, dm$ for all $A \in \Sigma$ is a vector measure. Moreover, $g \in L^1(m_f)$ if and only if $gf \in L^1(m)$ and in this case $\int g \, dm_f = \int gf \, dm$. 
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$$

Note that $L^\infty(\{|x^* m|\}) \subset L^1(m)$. In particular every measure of the type $|x^* m|$ is finite as $|x^* m|(\Omega) \leq \|x^*\| \cdot \|\chi_\Omega\|_m$.

- Given $f \in L^1(m)$, the set function $m_f: \Sigma \to E$ given by $m_f(A) = \int_A f \, dm$ for all $A \in \Sigma$ is a vector measure. Moreover, $g \in L^1(m_f)$ if and only if $gf \in L^1(m)$ and in this case $\int g \, dm_f = \int gf \, dm$. 