Distributional chaos for translation C_0-semigroups

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In mathematics you don't understand things. You just get used to them.

Johann von Neumann

1. C_0-semigroups of operators
2. Hypercyclicity and Devaney chaos for C_0-semigroups of operators
3. Distributional chaos for C_0-semigroups of operators
   - Discretizations
   - Translation C_0-semigroups

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Definition

Let X be an infinite-dimensional separable Banach space. A one-parameter family $\{T_t : X \to X \ ; \ t \geq 0\}$ is a *strongly continuous semigroup of operators*, from now on *C_0-semigroup*, if the following three conditions are satisfied

- $T_0 = I$.
- $T_t T_s = T_{t+s}$, for all $t, s \geq 0$.
- $\lim_{t \to s} T_t x = T_s x$, for each $x \in X$ and $s \geq 0$.

Definition (Admissible weight)

By an *admissible weight* function on $\mathbb{R}^+$ we mean a measurable function $\rho : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

1. $\rho(\tau) > 0$ for all $\tau \in \mathbb{R}^+$,
2. there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(\tau) \leq M e^{\omega t} \rho(t + \tau)$ for all $\tau \in \mathbb{R}^+$ and all $t > 0$.

Let $1 \leq p < \infty$ and an admissible weight function ρ on $\mathbb{R}^+$. We consider the separable Banach space of p-integrable functions (in the Lebesgue sense) $L^p_\rho(\mathbb{R}^+)$ as

$$\{f \in \mathcal{M}([0,+\infty[) : \|f\|_p < \infty\},$$

where

$$\|f\|_p = \left(\int_{[0,+\infty[} |f(s)|^p \rho(s) ds\right)^{1/p}.$$

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The *translation semigroup* defined by $(T_t f)(x) = f(x+t)$, $t, x \geq 0$, is a well-defined C_0-semigroup by the definition of admissible weight.

From now on X will denote an infinite-dimensional separable Banach space.

Definition

A C_0-semigroup $\mathcal{T} = \{T_t\}_{t \geq 0}$ on a Banach space X is *hypercyclic* if there are $x \in X$ whose orbit $\mathcal{O}(x, \mathcal{T}) := \{T_t x \ ; \ t \geq 0\}$ under $\mathcal{T}$ is dense in X. In such a case, x is called a *hypercyclic vector* for $\mathcal{T}$. We denote by $HC(\mathcal{T})$ the set of these vectors.

Definition

The set of *periodic* points of a C_0-semigroup $\mathcal{T} = \{T_t\}_{t \geq 0}$ is $Per(\mathcal{T}) := \{x \in X \ ; \ T_t x = x \text{ for some } t > 0\}$. Hence the C_0-semigroup $\mathcal{T}$ is said to be *chaotic* if it is hypercyclic and the set of periodic points $Per(\mathcal{T})$ is dense in X.

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Theorem (Desch et al.[2])

On $L^p_\rho(\mathbb{R}^+)$, the translation semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ is hypercyclic if and only if

$$\liminf_{t\to\infty} \rho(t) = 0.$$

Theorem (deLaubenfels, Emamirad [4])

We consider the translation C_0-semigroup on the space $X = L^p_\rho(\mathbb{R}^+)$, $1 \leq p < \infty$, for an admissible weight ρ. Then the following are equivalent:

- *The translation C_0-semigroup is chaotic on X.*
- $\int_0^\infty \rho(x)dx < \infty$.

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Theorem (Conejero-Müller-Peris [5])

Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a hypercyclic C_0-semigroup in $L(X)$, and let $x \in HC(\mathcal{T})$. Then $x \in HC(T_{t_0})$ for every $t_0 > 0$.

However we don't have an analogous result in the chaotic case.

Theorem (Bayart-Bermúdez [6])

There exists a C_0-semigroup $\{T_t\}_{t\geq 0}$ on a separable Hilbert space H and $t_0 \neq t_1$ such that T_{t_0} is chaotic and T_{t_1} is not chaotic.

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Definition

The *upper density* $\overline{\text{dens}}(\mathcal{K})$ of a set $\mathcal{K} \subset \mathbb{N}$ is defined by:

$$\overline{\text{dens}}(\mathcal{K}) := \limsup_{n\to\infty} \frac{|\mathcal{K} \cap \{1,\ldots,n\}|}{n}, \tag{4.1}$$

and the *lower density* $\underline{\text{dens}}(\mathcal{K})$ by:

$$\underline{\text{dens}}(\mathcal{K}) := \liminf_{n\to\infty} \frac{|\mathcal{K} \cap \{1,\ldots,n\}|}{n}. \tag{4.2}$$

The *upper density* $\overline{\text{Dens}}(A)$ of a Lebesgue measurable set $A \subset \mathbb{R}$ is defined by:

$$\overline{\text{Dens}}(A) := \limsup_{t\to\infty} \frac{\mu(A \cap [0,t])}{t}, \tag{4.3}$$

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Definition (Scrambled set)

An uncountable subset $S \subset X$ on a Banach space X is called a *scrambled set* for a C_0-semigroup $\mathcal{T} = \{T_t\}_{t \geq 0}$ if for any $f, g \in S$ with $f \neq g$ we have $\liminf_{t\to\infty} d(T_t f, T_t g) = 0$ and $\limsup_{t\to\infty} d(T_t f, T_t g) > 0$.

Definition (Distributional Chaos)

A C_0-semigroup of bounded linear operators $\{T_t\}_{t\geq 0}$ on X with a scrambled set S is *distributionally chaotic* on S if there is $\delta > 0$ so that for each $\varepsilon > 0$ and each pair $f, g \in S$ of distinct points we have

$$\limsup_{t\to\infty} \frac{\mu(\{s \leq t : d(T_s f, T_s g) > \delta\})}{t} = 1 \tag{4.5}$$

and

$$\limsup_{t\to\infty} \frac{\mu(\{s \leq t : d(T_s f, T_s g) < \varepsilon\})}{t} = 1. \tag{4.6}$$

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Theorem

Let $\mathcal{T} := \{T_t\}_{t \geq 0}$ be a C_0-semigroup. Then the following properties are equivalent:

1. *$\mathcal{T}$ is distributionally chaotic.*
2. *T_t is distributionally chaotic for each $t > 0$.*
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Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ *be the translation semigroup on* $X = L^p_\rho(\mathbb{R}^+)$. *The following are equivalent:*

1. *There exist* $f \in X$ *and* $\delta > 0$ *such that*

$$\underline{\text{Dens}}\{s \in \mathbb{R}^+ : \|T_s f\|_p < \delta\} = 0.$$

2. *There exists* $f \in X$ *such that, for every* $N > 0$,

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Theorem

The translation semigroup $\mathcal{T} = \{T_t\}_{t \geq 0}$ *is densely distributionally chaotic on* X *if we can find a measurable subset* $A \subset \mathbb{R}^+$ *such that* $\overline{\text{Dens}}(A) = 1$ *and* $\int_A \rho(s) ds < \infty$.

Example

For $k = 1, 2, \ldots$ *denote* $n_k = 2^{k^2}$ *and observe that*

$$n_1 + \ldots + n_k = 2 + 16 + \ldots + 2^{k^2} \leq 2^{k^2+1} \quad \textit{and so}$$

$$\frac{n_{k+1}}{n_1 + \ldots + n_{k+1}} \geq \frac{2^{(k+1)^2}}{2^{(k+1)^2} + 2^{k^2+1}} = \frac{1}{1 + 4^{-k}}. \tag{4.7}$$

Now we denote $m_k = \sum_{i=1}^{k} n_i$ *and define*

$$\rho(t) := \begin{cases} e^{-t}, & t \in [0, n_1[\\ e^{-t+m_k}, & t \in [m_k, m_{k+1}[\qquad \forall k \in \mathbb{N}. \end{cases}$$

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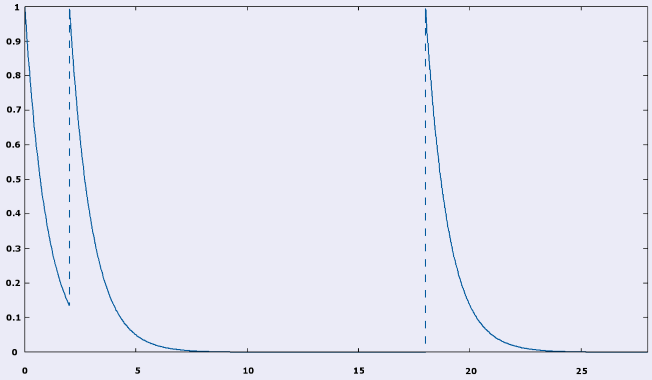

Example

Clearly we have $\int_{\mathbb{R}^+} \rho(t)dt = \infty$ *and therefore* $\mathcal{T}$ *on* $L^p_\rho(\mathbb{R}^+)$ *cannot be chaotic in the sense of Devaney. On the other hand we have that* $\int_A \rho(t)dt < \infty$. *It remains to show that A has upper density equal to 1. To this aim it is enough to prove that there is a sequence* $\{t_k\}_{k\in\mathbb{N}}$ *for which*

$$\lim_{k\to\infty} \frac{\mu(A\cap[0,t_k])}{t_k} = 1. \tag{4.7}$$

If we set $t_k = m_k$ *for every* $k \in \mathbb{N}$, *the limit above equals to* $\lim_{k\to\infty} \frac{n_k}{m_k}$. *Therefore we have that*

$$\lim_{k\to\infty} \frac{\mu(A\cap[0,t_k])}{t_k} = \lim_{k\to\infty} \frac{n_k}{m_k} = 1.$$

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Proposition

There exists $\mathcal{K} \subset \mathbb{N}$ *such that* $\overline{\text{dens}}(\mathcal{K}) = 1$ *and* $\sum_{k \in \mathcal{K}} \rho(k) < \infty$*, if and only if we can find an* $A \subset \mathbb{R}$ *such that* $\overline{\text{Dens}}(A) = 1$ *and* $\int_{t \in A} \rho(t) dt < \infty$*.*

Corollary

If we can find a subset $\mathcal{K} \subset \mathbb{N}$ *such that* $\overline{\text{dens}}(\mathcal{K}) = 1$ *and* $\sum_{k \in \mathcal{K}} \rho(k) < \infty$ *then the translation semigroup* $T = \{T_t\}_{t \geq 0}$ *is densely distributionally chaotic.*

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There exists $\mathcal{K} \subset \mathbb{N}$ such that $\overline{\text{dens}}(\mathcal{K}) = 1$ and $\sum_{k \in \mathcal{K}} \rho(k) < \infty$, if and only if we can find an $A \subset \mathbb{R}$ such that $\overline{\text{Dens}}(A) = 1$ and $\int_{t \in A} \rho(t) dt < \infty$.

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Theorem

Let $\rho : [0, +\infty[\longrightarrow \mathbb{R}^+$ be an admissible weight function such that the translation C_0-semigroup $\mathcal{T} = \{T_t\}_{t \geq 0}$ is distributionally chaotic on $L^p_\rho(\mathbb{R}^+)$, then for every sequence of weights $v = (v_n)_{n \in \mathbb{N}}$ such that there exist $0 < a < A < \infty$ with $a\rho(n-1) \leq v_n \leq A\rho(n)$, $n \in \mathbb{N}$, the backward shift B is distributionally chaotic on $\ell^p(v)$.

Theorem

Let $v = (v_n)_{n \in \mathbb{N}}$ be a sequence of positive weights such that the backward shift B is distributionally chaotic on $\ell^p(v)$, then for every admissible weight function ρ for which there are $0 < a < A < \infty$ satisfying $av_n \leq \rho(t) \leq Av_{n+1}$ for $t \in [n, n+1[$, the translation C_0-semigroup is distributionally chaotic on $L^p_\rho(\mathbb{R}^+)$.

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Sketch of Proof.

Let S be the scrambled set for B. For every $x = (x_0, x_1, x_2, \dots) \in S$ we can associate a function $f_x = \sum_{n=0}^{\infty} x_{n+2}\chi_{[n,n+1[}$, which verifies that T_1 acts on f_x as the backward shift does on the sequence x. Clearly, this function f_x is in $L^p_\rho(\mathbb{R}^+)$, since

$$\int_0^\infty |f(t)|^p \rho(t)dt \leq AM \sum_{n=0}^{\infty} |x_{n+2}|^p v_{n+2} < \infty,$$

where $\frac{v_n}{v_{n+1}} \leq M < \infty$ for all $n \in \mathbb{N}$ by the definition of positive weight.

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where $\dfrac{v_n}{v_{n+1}} \leq M < \infty$ for all $n \in \mathbb{N}$ by the definition of positive weight.
Let $x, y \in S$ and f_x, f_y be the corresponding elements in $L^p_\rho(\mathbb{R}^+)$. We have

$$d(B^{k+2}x, B^{k+2}y)^p \leq \frac{1}{a} d(T_1^k f_x, T_1^k f_y)^p \leq \frac{A}{a} d(B^{k+1}x, B^{k+1}y)^p.$$

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$$d(B^{k+2}x, B^{k+2}y)^p \leq \frac{1}{a} d(T_1^k f_x, T_1^k f_y)^p \leq \frac{A}{a} d(B^{k+1}x, B^{k+1}y)^p.$$

Hence

$$|\{k \leq n : d(T_1^k f, T_1^k g) < \delta' a^{1/p}\}| \leq |\{k \leq n+2 : d(B^k x, B^k y) < \delta'\}| .$$

And therefore exists a $\delta = \delta' a^{1/p} > 0$ such that for every $f_x, f_y \in \{f_x : x \in S\}$ with $f_x \neq f_y$ we have

$$\underline{\text{dens}}(\{k \in \mathbb{N} : d(T_1^k f_x, T_1^k f_y) < \delta\}) = 0.$$

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$$d(B^{k+2}x, B^{k+2}y)^p \leq \frac{1}{a} d(T_1^k f_x, T_1^k f_y)^p \leq \frac{A}{a} d(B^{k+1}x, B^{k+1}y)^p.$$

Finally, since

$$|\{k \leq n+1 : d(B^k x, B^k y) < \varepsilon\}| \leq |\{k \leq n : d(T_1^k f_x, T_1^k f_y) < A^{1/p}\varepsilon\}|,$$

and B is distributionally chaotic with respect to the scrambled set S, we get that

$$\overline{\text{dens}}(\{k \in \mathbb{N} : d(T_1^k f_x, T_1^k f_y) < \varepsilon\}) = 1, \quad \forall \varepsilon > 0$$

and therefore we conclude that T_1 is distributionally chaotic with a scrambled set $\{f_x : x \in S\}$. ⌟

Remark

Obviously there exist weights as the ones in the previous theorems. In the first theorem we can define the weights $v_n := \rho(n)$ for each $n \in \mathbb{N}$. In the second one we can take for instance the polygonal formed by the sequence v as an admissible weight function.

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