Abstract

In order to obtain extensions of the Banach Contraction Principle to the fuzzy context, several concepts of (fuzzy) contractivity have been given in the literature. In this talk we relate some different contractivity conditions which we state in the context of fuzzy metric spaces. Moreover, we introduce a new notion of contractivity in fuzzy metric spaces which is a particular case of a previous notion due to Mihet.
A fuzzy metric space is an ordered triple \((X, M, \ast)\) such that \(X\) is a (non-empty) set, \(\ast\) is a continuous \(t\)-norm and \(M\) is a fuzzy set on \(X \times X \times [0, \infty)\) satisfying the following conditions, for all \(x, y, z \in X, s, t > 0:\)

- (GV1) \(M(x, y, t) > 0;\)
- (GV2) \(M(x, y, t) = 1\) if and only if \(x = y;\)
- (GV3) \(M(x, y, t) = M(y, x, t);\)
- (GV4) \(M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s);\)
- (GV5) \(M(x, y, \_): [0, \infty] \rightarrow [0, 1]\) is continuous.

If \((X, M, \ast)\) is a fuzzy metric space, we will say that \((M, \ast)\) (or simply \(M\)) is a fuzzy metric on \(X\).
Some contractive conditions

Fuzzy metric space (George and Veeramani, 1994)

A fuzzy metric space is an ordered triple \((X, M, \ast)\) such that \(X\) is a (non-empty) set, \(\ast\) is a continuous \(t\)-norm and \(M\) is a fuzzy set on \(X \times X \times ]0, \infty[\) satisfying the following conditions, for all \(x, y, z \in X, s, t > 0:\)

\[
\begin{align*}
(GV1) & \quad M(x, y, t) > 0; \\
(GV2) & \quad M(x, y, t) = 1 \text{ if and only if } x = y; \\
(GV3) & \quad M(x, y, t) = M(y, x, t); \\
(GV4) & \quad M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s); \\
(GV5) & \quad M(x, y, \_ ) : ]0, \infty[ \mapsto ]0, 1] \text{ is continuous.}
\end{align*}
\]

If \((X, M, \ast)\) is a fuzzy metric space, we will say that \((M, \ast)\) (or simply \(M\)) is a fuzzy metric on \(X\).
A fuzzy metric space is an ordered triple \((X, M, \ast)\) such that \(X\) is a (non-empty) set, \(\ast\) is a continuous \(t\)-norm and \(M\) is a fuzzy set on \(X \times X \times ]0, \infty[\) satisfying the following conditions, for all \(x, y, z \in X\), \(s, t > 0\):

1. \((GV1)\) \(M(x, y, t) > 0\);
2. \((KM1)\) \(M(x, y, 0) = 0\);
3. \((GV2)\) \(M(x, y, t) = 1\) if and only if \(x = y\);
4. \((KM2)\) \(M(x, y, t) = 1\) for all \(t > 0\) if and only if \(x = y\);
5. \((GV3)\) \(M(x, y, t) = M(y, x, t)\);
6. \((GV4)\) \(M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)\);
7. \((GV5)\) \(M(x, y, \_): ]0, \infty[ \rightarrow [0, 1]\) is continuous.
8. \((KM5)\) \(M(x, y, \_): [0, \infty[ \rightarrow [0, 1]\) is left continuous.

We will refer to these fuzzy metric spaces as \(KM\)-fuzzy metric spaces.
A fuzzy metric space is an ordered triple $(X, M, \ast)$ such that $X$ is a (non-empty) set, $\ast$ is a continuous $t$-norm and $M$ is a fuzzy set on $X \times X \times [0, \infty]$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$;
(KM1) $M(x, y, 0) = 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y$;
(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$;
(GV5) $M(x, y, \_ : [0, \infty] \to [0, 1]$ is continuous;
(KM5) $M(x, y, \_ : [0, \infty] \to [0, 1]$ is left continuous.

We will refer to these fuzzy metric spaces as $KM$-fuzzy metric spaces.
A fuzzy metric space is an ordered triple \((X, M, *)\) such that \(X\) is a (non-empty) set, \(*\) is a continuous \(t\)-norm and \(M\) is a fuzzy set on \(X \times X \times ]0, \infty[\) satisfying the following conditions, for all \(x, y, z \in X, s, t > 0\):

(GV1) \(M(x, y, t) > 0\);
(KM1) \(M(x, y, 0) = 0\);
(GV2) \(M(x, y, t) = 1\) if and only if \(x = y\);
(KM2) \(M(x, y, t) = 1\) for all \(t > 0\) if and only if \(x = y\);
(GV3) \(M(x, y, t) = M(y, x, t)\);
(GV4) \(M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)\);
(GV5) \(M(x, y, _) : ]0, \infty[ \rightarrow ]0, 1]\) is continuous.
(KM5) \(M(x, y, _) : [0, \infty[ \rightarrow [0, 1]\) is left continuous.

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We will refer to these fuzzy metric spaces as \(KM\)-fuzzy metric spaces.
Stationary fuzzy metric (Gregori and Romaguera, 2004)

A fuzzy metric $M$ on $X$ is said to be stationary if $M$ does not depend on $t$, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.

From now on $(X, M, *)$ is a fuzzy metric space and $f$ is a self mapping of $X$. 
Stationary fuzzy metric (Gregori and Romaguera, 2004)

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From now on $(X, M, \ast)$ is a fuzzy metric space and $f$ is a self mapping of $X$. 
Contractivity in metric spaces

Recall that a self-mapping \( f \) in a metric space \((X, d)\) is said to be contractive if there exists \( k \in ]0, 1[ \) such that

\[
d(f(x), f(y)) \leq k \, d(x, y) \quad \text{for all } x, y \in X \tag{1}
\]

G-contractive mapping (Grabiec, 1989)

A mapping \( f \) is said to be \( G \)-contractive if there exists \( k \in ]0, 1[ \) such that for all \( x, y \in X, t > 0 \)

\[
M(f(x), f(y), kt) \geq M(x, y, t) \tag{2}
\]

Notice that this definition is not appropriate when \( M \) is a stationary fuzzy metric because in this case it becomes

\[
M(f(x), f(y)) \geq M(x, y) \quad \text{for all } x, y \in X \quad \tag{3}
\]
Preliminaries

Some contractive conditions

Contractivity in metric spaces

Recall that a self-mapping $f$ in a metric space $(X, d)$ is said to be contractive if there exists $k \in ]0, 1[$ such that

$$d(f(x), f(y)) \leq k \, d(x, y) \text{ for all } x, y \in X$$

(1)

G-contractive mapping (Grabiec, 1989)

A mapping $f$ is said to be $G$-contractive if there exists $k \in ]0, 1[$ such that for all $x, y \in X$, $t > 0$

$$M(f(x), f(y), kt) \geq M(x, y, t)$$

(2)

Notice that this definition is not appropriate when $M$ is a stationary fuzzy metric because in this case it becomes

$$M(f(x), f(y)) \geq M(x, y) \text{ for all } x, y \in X$$

(3)
Contractivity in metric spaces

Recall that a self-mapping $f$ in a metric space $(X, d)$ is said to be contractive if there exists $k \in ]0, 1[$ such that

$$d(f(x), f(y)) \leq k d(x, y) \text{ for all } x, y \in X \tag{1}$$

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$$M(f(x), f(y)) \geq M(x, y) \text{ for all } x, y \in X \tag{3}$$
Fuzzy metrics deduced from a metric

Let \((X, d)\) be a metric space and let \((M, \ast)\) be a fuzzy metric on \(X\). We will say that \(M\) is deduced (explicitly) from \(d\) if in the formulation of \(M\) appears explicitly the metric \(d\) (that is, \(M\) is defined using \(d\)).
Standard fuzzy metric space (George and Veeramani, 1994)

Let \((X, d')\) be a metric space and let \(M_d\) a function on \(X^2 \times ]0, \infty[\) defined by

\[
M_d(x, y, t) = \frac{t}{t + d(x, y)}
\]

Then \((M_d, \cdot)\) is called the **standard fuzzy metric** induced by \(d\).

\(M_2\) fuzzy metric (George and Veeramani, 1994)

Let \((X, d')\) be a metric space. Then \((X, M_2, \cdot)\) is a fuzzy metric space where

\[
M_2(x, y, t) = e^{-\frac{d(x, y)}{t}}
\]

\(M_1\) fuzzy metric space

Let \((X, d')\) be a metric space with \(d(x, y) \leq 1\) for all \(x, y \in X\). Then \((X, M_1, \mathcal{L})\) is a fuzzy metric space where

\[
M_1(x, y, t) = 1 - \frac{d(x, y)}{1 + t}
\]
Preliminaries

Some contractive conditions

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Preliminaries

Some contractive conditions

Contractivity in metric spaces

Recall that a self-mapping $f$ in a metric space $(X, d)$ is said to be contractive if there exists $k \in ]0, 1[$ such that

$$d(f(x), f(y)) \leq k d(x, y) \text{ for all } x, y \in X$$  \hspace{1cm} (4)

Contractivity in fuzzy metric spaces deduced from metric spaces

We focus our attention in conditions of contractivity for fuzzy metric spaces deduced from metric spaces and we are interested in finding the corresponding expressions of the contractivity in the above fuzzy metrics. These expressions motivate some of the well-known fuzzy contractive conditions appeared in the literature.

S. Morillas - A. Sapena

Contractivity in fuzzy metric spaces deduced from metrics
Preliminaries Some contractive conditions

Contractivity in metric spaces

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$$d(f(x), f(y)) \leq k d(x, y) \text{ for all } x, y \in X$$

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Contractivity in fuzzy metric spaces deduced from metric spaces

We focus our attention in conditions of contractivity for fuzzy metric spaces deduced from metric spaces and we are interested in finding the corresponding expressions of the contractivity in the above fuzzy metrics. These expressions motivate some of the well-known fuzzy contractive conditions appeared in the literature.
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Let \((X, d)\) be a metric space and let \(M_d\) a function on \(X^2 \times ]0, \infty[\) defined by

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M_d(x, y, t) = \frac{t}{t + d(x, y)}
\]

Then \((M_d, \cdot)\) is called the standard fuzzy metric induced by \(d\).

Contractivity in \(M_d\)

A self-mapping \(f\) on a fuzzy metric space \((X, M_d, \cdot)\) is GS-contractive if there exists \(k \in ]0, 1[\) satisfying for all \(x, y \in X\) and \(t > 0\)

\[
\frac{1}{M_d(f(x), f(y), t)} - 1 \leq k \left( \frac{1}{M_d(x, y, t)} - 1 \right)
\]

GS-contractive mapping (Gregori and Sapena, 2002)

A self mapping \(f\) on a fuzzy metric space \((X, M, \ast)\) is GS-contractive if there exists \(k \in ]0, 1[\) satisfying for all \(x, y \in X\) and \(t > 0\)

\[
\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left( \frac{1}{M(x, y, t)} - 1 \right) \quad (5)
\]
**Standard fuzzy metric space (George and Veeramani, 1994)**

Let \((X, d)\) be a metric space and let \(M_d\) a function on \(X^2 \times ]0, \infty[\) defined by

\[
M_d(x, y, t) = \frac{t}{t + d(x, y)}
\]

Then \((M_d, \cdot)\) is called the *standard fuzzy metric* induced by \(d\).

**Contractivity in \(M_d\)**

A self-mapping \(f\) on a fuzzy metric space \((X, M_d, \cdot)\) is GS-contractive if there exists \(k \in ]0, 1[\) satisfying for all \(x, y \in X\) and \(t > 0\)

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\frac{1}{M_d(f(x), f(y), t)} - 1 \leq k \left( \frac{1}{M_d(x, y, t)} - 1 \right)
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**GS-contractive mapping (Gregori and Sapena, 2002)**

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\frac{1}{M_d(f(x), f(y), t)} - 1 \leq k \left( \frac{1}{M_d(x, y, t)} - 1 \right)
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GS-contractive mapping (Gregori and Sapena, 2002)

A self mapping \(f\) on a fuzzy metric space \((X, M, \ast)\) is GS-contractive if there exists \(k \in ]0, 1[\) satisfying for all \(x, y \in X\) and \(t > 0\)

\[
\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left( \frac{1}{M(x, y, t)} - 1 \right)
\]

(5)
Proposition

Let \((X, d')\) be a metric space and consider the corresponding standard fuzzy metric space \((X, M_d, \cdot)\). Let \(f : X \to X\) be a mapping. The following are equivalent:

(i) \(f\) is \(d\)-contractive with constant \(k\).
(ii) \(f\) is \(G\)-contractive with constant \(k\).
(iii) \(f\) is \(GS\)-contractive with constant \(k\).
GS-contractive mapping (Gregori and Sapena, 2002)

A self mapping \( f \) on a fuzzy metric space \((X, M, \ast)\) is GS-contractive if there exists \( k \in ]0, 1[\) satisfying for all \( x, y \in X \) and \( t > 0 \)

\[
\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left( \frac{1}{M(x, y, t)} - 1 \right)
\] (6)

(Radu, 2002)

\[
M(f(x), f(y), t) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))}
\] (7)

Notice that condition (7) is more convenient than (6) because it remains valid for \( KM \)-fuzzy metric spaces in which the value 0 for \( M(x, y, t) \) is possible.
GS-contractive mapping (Gregori and Sapena, 2002)

A self mapping $f$ on a fuzzy metric space $(X, M, \ast)$ is GS-contractive if there exists $k \in ]0, 1[$ satisfying for all $x, y \in X$ and $t > 0$

\[
\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left( \frac{1}{M(x, y, t)} - 1 \right)
\]  

(6)

(Radu, 2002)

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M(f(x), f(y), t) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))}
\]

(7)

Notice that condition (7) is more convenient than (6) because it remains valid for $KM$-fuzzy metric spaces in which the value 0 for $M(x, y, t)$ is possible.
The following contractive condition was introduced by Radu in order to obtain a Banach fixed point theorem in the context of $KM$-fuzzy metric spaces strict $GS$-contractive mapping (Radu, 2002)

A self mapping $f$ on a fuzzy metric space $(X, M, \ast)$ is strict $GS$-contractive:

$$M(f(x), f(y), kt) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))}$$

where $k$ is a fixed constant in $]0, 1[$.

It is well-known that Radu’s strict $GS$-contractivity implies $GS$-contractivity. Now, in a standard fuzzy metric space they are equivalent.

Proposition

Let $f$ be a mapping in the standard fuzzy metric space $(X, M_d, \cdot)$. Then $f$ is strictly $GS$-contractive if and only if $f$ is $GS$-contractive.
The following contractive condition was introduced by Radu in order to obtain a Banach fixed point theorem in the context of $KM$-fuzzy metric spaces

**strict GS-contractive mapping (Radu, 2002)**

A self mapping $f$ on a fuzzy metric space $(X, M, *)$ is strict GS-contractive:

$$M(f(x), f(y), kt) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))}$$  \hfill (8)

where $k$ is a fixed constant in $]0, 1[$.

It is well-known that Radu’s strict GS-contractivity implies GS-contractivity. Now, in a standard fuzzy metric space they are equivalent.

**Proposition**

Let $f$ be a mapping in the standard fuzzy metric space $(X, M_d, \cdot)$. Then $f$ is strictly GS-contractive if and only if $f$ is GS-contractive.
$M_1$ fuzzy metric space

Let $(X, d)$ be a metric space with $d(x, y) \leq 1$ for all $x, y \in X$. Then $(X, M_1, \mathcal{L})$ is a fuzzy metric space where

$$M_1(x, y, t) = 1 - \frac{d(x, y)}{1 + t}$$

Contractivity in $M_1$

A self-mapping $f$ on the fuzzy quasi-metric space $(X, M_1, \mathcal{L})$ is $RT$-contractive if there exists $k \in ]0, 1[$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

$$M_1(f(x), f(y), t) \geq 1 - k + k M_1(x, y, t)$$

$RT$-contractive mapping (Romaguera and Tirado, 2009)

A self-mapping $f$ on a fuzzy quasi-metric space $(X, M, \ast)$ is $RT$-contractive if there exists $k \in ]0, 1[$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

$$M(f(x), f(y), t) \geq 1 - k + k M(x, y, t)$$  (9)
\( M_1 \) fuzzy metric space

Let \((X, d)\) be a metric space with \(d(x, y) \leq 1\) for all \(x, y \in X\). Then \((X, M_1, \mathcal{L})\) is a fuzzy metric space where

\[
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\]

Contractivity in \(M_1\)

A self-mapping \(f\) on the fuzzy quasi-metric space \((X, M_1, \mathcal{L})\) is \(RT\)-contractive if there exists \(k \in ]0, 1[\) such that for all \(x, y \in X\) and \(t > 0\) it is satisfied

\[
M_1(f(x), f(y), t) \geq 1 - k + k M_1(x, y, t)
\]

\(RT\)-contractive mapping (Romaguera and Tirado, 2009)

A self-mapping \(f\) on a fuzzy quasi-metric space \((X, M, \ast)\) is \(RT\)-contractive if there exists \(k \in ]0, 1[\) such that for all \(x, y \in X\) and \(t > 0\) it is satisfied

\[
M(f(x), f(y), t) \geq 1 - k + k M(x, y, t)
\]
**$M_1$ fuzzy metric space**

Let $(X, d)$ be a metric space with $d(x, y) \leq 1$ for all $x, y \in X$. Then $(X, M_1, \mathcal{L})$ is a fuzzy metric space where

$$M_1(x, y, t) = 1 - \frac{d(x, y)}{1 + t}$$

**Contractivity in $M_1$**

A self-mapping $f$ on the fuzzy quasi-metric space $(X, M_1, \mathcal{L})$ is $RT$-contractive if there exists $k \in ]0, 1[$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

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**$RT$-contractive mapping (Romaguera and Tirado, 2009)**

A self-mapping $f$ on a fuzzy quasi-metric space $(X, M, \ast)$ is $RT$-contractive if there exists $k \in ]0, 1[$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

$$M(f(x), f(y), t) \geq 1 - k + k M(x, y, t)$$  \hspace{1cm} (9)
Proposition
Let $f : X \to X$ be a mapping. Then $f$ is contractive in $(X, d)$ with constant $k$ if and only if $f$ is $RT$-contractive in $(X, M_1, \mathcal{L})$ with constant $k$. 
The authors showed that an $RT$-contractive mapping is $GS$-contractive.

**Proposition**

Let $(X, M, *)$ be a fuzzy metric space and let $f : X \rightarrow X$ be a self-mapping. If $f$ is $RT$-contractive then $f$ is $GS$-contractive.

The converse is not true.

**Example**

Consider the standard fuzzy metric space induced by the real line endowed with the usual metric. Let $f : X \rightarrow X$ defined by $f(x) = \frac{x}{2}$. It is well-known that $f$ is contractive and so it is $GS$-contractive. Now, let $k \in ]0, 1[$. Choice $y > 0$ such that $1 + \frac{y}{2} < 1 - k$, then $M_d(f(0), f(y), 1) = \frac{1}{1 + |\frac{y}{2} - 0|} < 1 - k$ and, in consequence, $f$ is not $RT$-contractive for any $k \in ]0, 1[$.
The authors showed that an \( RT \)-contractive mapping is \( GS \)-contractive.

**Proposition**

Let \( (X, M, \ast) \) be a fuzzy metric space and let \( f : X \to X \) be a self-mapping. If \( f \) is \( RT \)-contractive then \( f \) is \( GS \)-contractive.

The converse is not true.

**Example**

Consider the standard fuzzy metric space induced by the real line endowed with the usual metric. Let \( f : X \to X \) defined by \( f(x) = \frac{x}{2} \). It is well-known that \( f \) is contractive and so it is \( GS \)-contractive. Now, let \( k \in ]0, 1[ \). Choice \( y > 0 \) such that \( \frac{1}{1 + \frac{y}{2}} < 1 - k \), then \( M_d(f(0), f(y), 1) = \frac{1}{1 + |\frac{y}{2} - 0|} < 1 - k \) and, in consequence, \( f \) is not \( RT \)-contractive for any \( k \in ]0, 1[ \).
strict $RT$ contractive condition

Before, in [11], Sherwood introduced a concept of contractivity in the context of $PM$-spaces that in our terminology is called strict $RT$-contractive condition which is written as

$$M(f(x), f(y), kt) \geq 1 - k + kM(x, y, t)$$  \hspace{1cm} (10)

where $k$ is a fixed constant in $]0, 1[$.
RT-contraction is weaker than the given by Sherwood.

**Proposition**

Let \((X, d')\) be a metric space and consider the fuzzy metric space \((X, M_1, \mathcal{L})\). Then a mapping \(f : X \rightarrow X\) is RT-contractive (with constant \(k\)) if and only if it is strict RT-contractive (with constant \(\sqrt{k}\)).
D1-contractive mapping (Mihet, 2007)

Let $\Psi$ be the class of continuous increasing functions $\varphi : [0, 1] \rightarrow [0, 1]$ such that $\varphi(z) > z$ for all $z \in ]0, 1[$. A mapping $f$ is said to be D1-contractive for $\varphi$ if there exists $\varphi \in \Psi$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

$$M(f(x), f(y), t) \geq \varphi(M(x, y, t)) \quad (11)$$

Notice that the class of GS-contractions and RT-contractions belong to this class of D1-contractions.

In fact, if $f$ is GS-contractive for a constant $k \in ]0, 1[$ then it is D1-contractive by taking

$$\varphi_k(z) = \frac{z}{z + k(1 - z)}, \quad z \in [0, 1].$$

Also, if $f$ is RT-contractive it is D1-contractive by taking

$$\varphi^k(z) = 1 - k + kz, \quad z \in [0, 1].$$

It is easy to see that $\varphi^k \geq \varphi_k$.
Some contractive conditions

**D1-contractive mapping (Mihet, 2007)**

Let $\Psi$ be the class of continuous increasing functions $\varphi : [0, 1] \to [0, 1]$ such that $\varphi(z) > z$ for all $z \in ]0, 1[$. A mapping $f$ is said to be $D1$-contractive for $\varphi$ if there exists $\varphi \in \Psi$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

$$M(f(x), f(y), t) \geq \varphi(M(x, y, t))$$

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Notice that the class of $GS$-contractions and $RT$-contractions belong to this class of $D1$-contractions.

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Let \( \Psi \) be the class of continuous increasing functions \( \varphi : [0, 1] \rightarrow [0, 1] \) such that \( \varphi(z) > z \) for all \( z \in ]0, 1[ \). A mapping \( f \) is said to be \( D1 \)-contractive for \( \varphi \) if there exists \( \varphi \in \Psi \) such that for all \( x, y \in X \) and \( t > 0 \) it is satisfied

\[
M(f(x), f(y), t) \geq \varphi(M(x, y, t))
\]  

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In fact, if \( f \) is \( GS \)-contractive for a constant \( k \in ]0, 1[ \) then it is \( D1 \)-contractive by taking

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Also, if \( f \) is \( RT \)-contractive it is \( D1 \)-contractive by taking

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\varphi^k(z) = 1 - k + kz, \quad z \in [0, 1].
\]

It is easy to see that \( \varphi^k \geq \varphi_k \)
D1-contractive mapping (Mihet, 2007)

Let $\Psi$ be the class of continuous increasing functions $\varphi : [0, 1] \rightarrow [0, 1]$ such that $\varphi(z) > z$ for all $z \in ]0, 1[. A mapping $f$ is said to be D1-contractive for $\varphi$ if there exists $\varphi \in \Psi$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

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It is easy to see that $\varphi^k \geq \varphi_k$
**$M_2$ fuzzy metric (George and Veeramani, 1994)**

Let $(X, d)$ be a metric space. Then $(X, M_2, \cdot)$ is a fuzzy metric space where

$$M_2(x, y, t) = e^{-\frac{d(x, y)}{t}}$$

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**$M_2$-contractive mapping**

Let $(X, M, *)$ be a fuzzy metric space and let $f : X \to X$ be a mapping. We will say that $f$ is $M_2$-contractive if there exists $k \in ]0, 1[$ such that

$$M(f(x), f(y), t) \geq (M(x, y, t))^k$$

In the same way $f$ is said to be strict $M_2$-contractive if

$$M(f(x), f(y), kt) \geq (M(x, y, t))^k$$

Notice that this is a particular case of Mihet's $D1$-contractivity with $\varphi(t) = t^k$. 
$M_2$ fuzzy metric (George and Veeramani, 1994)

Let $(X, d)$ be a metric space. Then $(X, M_2, \cdot)$ is a fuzzy metric space where

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$M_2$-contractive mapping

Let $(X, M, \ast)$ be a fuzzy metric space and let $f : X \rightarrow X$ be a mapping. We will say that $f$ is $M_2$-contractive if there exists $k \in ]0, 1[$ such that

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Notice that this is a particular case of Mihet's $D1$-contractivity with $\varphi(t) = t^k$. 
Proposition

Let \((X, d)\) be a metric space and consider the fuzzy metric space \((X, M_2, \cdot)\). Let \(f : X \to X\) be a mapping. Then:

(i) \(f\) is contractive if and only if it is \(M_2\)-contractive.

(ii) \(f\) is contractive for the constant \(k < 1\) then it is strict \(M_2\)-contractive for the constant \(\sqrt{k}\).


