Contractivity in fuzzy metric spaces deduced from metrics

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Abstract

In order to obtain extensions of the Banach Contraction Principle to the fuzzy context, several concepts of (fuzzy) contractivity have been given in the literature. In this talk we relate some different contractivity conditions which we state in the context of fuzzy metric spaces. Moreover, we introduce a new notion of contractivity in fuzzy metric spaces which is a particular case of a previous notion due to Mihet.

Fuzzy metric space (George and Veeramani, 1994)

A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

- (GV1) $M(x, y, t) > 0$;
- (GV2) $M(x, y, t) = 1$ if and only if $x = y$;
- (GV3) $M(x, y, t) = M(y, x, t)$;
- (GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
- (GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ (or simply M) is a *fuzzy metric* on X.

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A fuzzy metric M on X is said to be *stationary* if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant.
In this case we write $M(x, y)$ instead of $M(x, y, t)$.

From now on $(X, M, *)$ is a fuzzy metric space and f is a self mapping of X.

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Contractivity in metric spaces

Recall that a self-mapping f in a metric space (X, d) is said to be contractive if there exists $k \in]0, 1[$ such that

$$d(f(x), f(y)) \leq k\, d(x, y) \text{ for all } x, y \in X \tag{1}$$

G-contractive mapping (Grabiec, 1989)

A mapping f is said to be G-contractive if there exists $k \in]0, 1[$ such that for all $x, y \in X$, $t > 0$

$$M(f(x), f(y), kt) \geq M(x, y, t) \tag{2}$$

Notice that this definition is not appropriate when M is a stationary fuzzy metric because in this case it becomes

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Fuzzy metrics deduced from a metric

Let (X, d) be a metric space and let $(M, *)$ be a fuzzy metric on X. We will say that M is deduced (explicitly) from d if in the formulation of M appears explicitly the metric d (that is, M is defined using d).

Standard fuzzy metric space (George and Veeramani, 1994)

Let (X, d) be a metric space and let M_d a function on $X^2 \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(M_d, \cdot)$ is called the *standard fuzzy metric* induced by d.

M_2 fuzzy metric (George and Veeramani, 1994)

Let (X, d) be a metric space. Then $(X, M_2, \cdot)$ is a fuzzy metric space where

$$M_2(x, y, t) = e^{-\frac{d(x,y)}{t}}$$

M_1 fuzzy metric space

Let (X, d) be a metric space with $d(x, y) \leq 1$ for all $x, y \in X$. Then $(X, M_1, \mathfrak{L})$ is a fuzzy metric space where

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Contractivity in fuzzy metric spaces deduced from metric spaces

We focus our attention in conditions of contractivity for fuzzy metric spaces deduced from metric spaces and we are interested in finding the corresponding expressions of the contractivity in the above fuzzy metrics. These expressions motivate some of the well-known fuzzy contractive conditions appeared in the literature.

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Proposition

Let (X, d) be a metric space and consider the corresponding standard fuzzy metric space $(X, M_d, \cdot)$. Let $f : X \to X$ be a mapping. The following are equivalent:

(i) f is d-contractive with constant k.

(ii) f is G-contractive with constant k.

(iii) f is GS-contractive with constant k.

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(Radu, 2002)

$$M(f(x), f(y), t) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))} \qquad (7)$$

Notice that condition (7) is more convenient than (6) because it remains valid for *KM*-fuzzy metric spaces in which the value 0 for $M(x, y, t)$ is possible.

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The following contractive condition was introduced by Radu in in order to obtain a Banach fixed point theorem in the context of *KM*-fuzzy metric spaces

strict *GS*-contractive mapping (Radu, 2002)

A self mapping f on a fuzzy metric space $(X, M, *)$ is strict *GS*-contractive:

$$M(f(x), f(y), kt) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))} \tag{8}$$

where k is a fixed constant in $]0, 1[$.

It is well-known that Radu's strict *GS*-contractivity implies *GS*-contractivity. Now, in a standard fuzzy metric space they are equivalent.

Proposition

Let f be a mapping in the standard fuzzy metric space $(X, M_d, \cdot)$. Then f is strictly *GS*-contractive if and only if f is *GS*-contractive.

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M_1 fuzzy metric space

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$$M_1(x, y, t) = 1 - \frac{d(x, y)}{1 + t}$$

Contractivity in M_1

A self-mapping f on the fuzzy quasi-metric space $(X, M_1, \mathfrak{L})$ is RT-contractive if there exists $k \in]0, 1[$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

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Proposition

Let $f : X \to X$ be a mapping. Then f is contractive in (X, d) with constant k if and only if f is RT-contractive in $(X, M_1, \mathfrak{L})$ with constant k.

The authors showed that an *RT*-contractive mapping is *GS*-contractive.

Proposition

Let $(X, M, *)$ be a fuzzy metric space and let $f : X \to X$ be a self-mapping. If f is *RT*-contractive then f is *GS*-contractive.

The converse is not true.

Example

Consider the standard fuzzy metric space induced by the real line endowed with the usual metric. Let $f : X \to X$ defined by $f(x) = \dfrac{x}{2}$. It is well-known that f is contractive and so it is *GS*-contractive. Now, let $k \in]0,1[$. Choice $y > 0$ such that $\dfrac{1}{1+\frac{y}{2}} < 1-k$, then $M_d(f(0), f(y), 1) = \dfrac{1}{1+|\frac{y}{2}-0|} < 1-k$ and, in consequence, f is not *RT*-contractive for any $k \in]0,1[$.

The authors showed that an RT-contractive mapping is GS-contractive.

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strict *RT* contractive condition

Before, in [11], Sherwood introduced a concept of contractivity in the context of *PM*-spaces that in our terminology is called strict *RT*-contractive condition which is written as

$$M(f(x), f(y), kt) \geq 1 - k + kM(x, y, t) \tag{10}$$

where k is a fixed constant in $]0, 1[$.

RT-contraction is weaker than the given by Sherwood.

Proposition

Let (X, d) be a metric space and consider the fuzzy metric space $(X, M_1, \mathfrak{L})$. Then a mapping $f : X \to X$ is RT-contractive (with constant k) if and only if it is strict RT-contractive (with constant $\sqrt{k}$).

$D1$-contractive mapping (Mihet, 2007)

Let Ψ be the class of continuous increasing functions $\varphi : [0,1] \to [0,1]$ such that $\varphi(z) > z$ for all $z \in]0,1[$. A mapping f is said to be $D1$-contractive for φ if there exists $\varphi \in \Psi$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

$$M(f(x), f(y), t) \geq \varphi(M(x,y,t)) \quad (11)$$

Notice that the class of GS-contractions and RT-contractions belong to this class of $D1$-contractions.

In fact, if f is GS-contractive for a constant $k \in]0,1[$ then it is $D1$-contractive by taking

$$\varphi_k(z) = \frac{z}{z + k(1-z)}, \qquad z \in [0,1].$$

Also, if f is RT-contractive it is $D1$-contractive by taking

$$\varphi^k(z) = 1 - k + kz, \qquad z \in [0,1].$$

It is easy to see that $\varphi^k \geq \varphi_k$

$D1$-contractive mapping (Mihet, 2007)

Let Ψ be the class of continuous increasing functions $\varphi : [0,1] \to [0,1]$ such that $\varphi(z) > z$ for all $z \in]0,1[$. A mapping f is said to be $D1$-contractive for φ if there exists $\varphi \in \Psi$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

$$M(f(x), f(y), t) \geq \varphi(M(x, y, t)) \tag{11}$$

Notice that the class of GS-contractions and RT-contractions belong to this class of $D1$-contractions.

In fact, if f is GS-contractive for a constant $k \in]0,1[$ then it is $D1$-contractive by taking

$$\varphi_k(z) = \frac{z}{z + k(1-z)}, \qquad z \in [0,1].$$

Also, if f is RT-contractive it is $D1$-contractive by taking

$$\varphi^k(z) = 1 - k + kz, \qquad z \in [0,1].$$

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M_2 fuzzy metric (George and Veeramani, 1994)

Let (X, d) be a metric space. Then $(X, M_2, \cdot)$ is a fuzzy metric space where

$$M_2(x, y, t) = e^{-\frac{d(x,y)}{t}}$$

M_2-contractive mapping

Let $(X, M, *)$ be a fuzzy metric space and let $f : X \to X$ be a mapping. We will say that f is M_2-contractive if there exists $k \in]0, 1[$ such that

$$M(f(x), f(y), t) \geq (M(x, y, t))^k$$

In the same way f is said to be strict M_2-contractive if

$$M(f(x), f(y), kt) \geq (M(x, y, t))^k$$

Notice that this is a particular case of Mihet's $D1$-contractivity with $\varphi(t) = t^k$.

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Proposition

Let (X, d) be a metric space and consider the fuzzy metric space $(X, M_2, \cdot)$. Let $f : X \to X$ be a mapping. Then:

(i) f is contractive if and only if it is M_2-contractive.

(ii) f is contractive for the constant $k < 1$ then it is strict M_2-contractive for the constant $\sqrt{k}$.

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