Compact subsets in the fuzzy number space

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$$[u]^\lambda := \{\, x \in \mathbb{R} \mid u(x) \geq \lambda \,\}, \quad \lambda \in]0,1],$$

$$[u]^0 := \mathrm{cl}_{\mathbb{R}}\{\, x \in \mathbb{R} \mid u(x) > 0 \,\}.$$

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- u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ with $u(x_0) = 1$;
- u is convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0,1]$;
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$$\tilde{r}(t) := \begin{cases} 1 & \text{if } t = r, \\ 0 & \text{if } t \neq r. \end{cases}$$

Theorem (Goetschel and Voxman (1986)) Let $u \in \mathbb{E}^1$ and $[u]^\lambda = [u^-(\lambda), u^+(\lambda)]$, $\lambda \in [0,1]$. Then the pair of functions $u^-(\lambda)$ and $u^+(\lambda)$ has the following properties:

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Conversely, if a pair of functions $\alpha(\lambda)$ and $\beta(\lambda)$ satisfy the above conditions (i)-(iv), then there exists a unique $u \in \mathbb{E}^1$ such that $[u]^\lambda = [\alpha(\lambda), \beta(\lambda)]$ for each $\lambda \in [0,1]$.

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which is a metric on $\mathbb{E}^1$. It is called the supremum metric on $\mathbb{E}^1$, and $(\mathbb{E}^1, d_\infty)$ is a complete metric space.

J-X. Fang and Q-Y. Xue, *Some properties of the space of fuzzy-valued continuous functions on a compact set*, Fuzzy Sets and Systems 160 (2009) 1620–1631.

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Consider the sequence of fuzzy numbers $M = \{(u_n^+, u_n^-)\}$ where

$$u_n^+(\lambda) = \begin{cases} 1 & \text{if } \lambda \in [0, \frac{1}{2}], \\ \frac{1}{2} & \text{if } \lambda \in]\frac{1}{2}, \frac{1}{2} + \frac{1}{n}], \\ 0 & \text{if } \lambda \in]\frac{1}{2} + \frac{1}{n}, 1] \end{cases}$$

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It is straightforward to check that $\{u_n^+\}$ is left-equicontinuous. Moreover, each subsequence of $\{u_n^+\}$ pointwise converges to the function $u(\lambda) = 1$ if $0 \leq \lambda \leq \frac{1}{2}$ and $u(\lambda) = 0$ if $\frac{1}{2} < \lambda \leq 1$.

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It is straightforward to check that $\{u_n^+\}$ is left-equicontinuous. Moreover, each subsequence of $\{u_n^+\}$ pointwise converges to the function $u(\lambda) = 1$ if $0 \leq \lambda \leq \frac{1}{2}$ and $u(\lambda) = 0$ if $\frac{1}{2} < \lambda \leq 1$. Moreover, each subsequence of $\{u_n^+\}$ pointwise converges to the function $u(\lambda) = 1$ if $0 \leq \lambda \leq \frac{1}{2}$ and $u(\lambda) = 0$ if $\frac{1}{2} < \lambda \leq 1$. Since this convergence is not uniform, the sequence M is a noncompact closed subset of $(\mathbb{E}^1, d_\infty)$.

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(ii) $\{ u^+ \mid u \in M \}$ and $\{ u^- \mid u \in M \}$ are left-equicontinuous on $]0, 1]$ and almost-right-equicontinuous on $[0, 1[$.

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Any bounded sequence of monotonic real-valued functions on $[0, 1]$ contains a pointwise convergent subsequence.

For Further Reading

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For Further Reading

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Thanks for your attention !!