Characterising perturbations of isomorphisms between operator algebras through spectral inclusions

A. R. Villena
joint work with J. Alaminos and J. Extremera

Departamento de Análisis Matemático
Universidad de Granada

Trimestre Temático de Análisis Funcional
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Universidad Politécnica de Valencia

Part I

Seminal results: characterising multiplicativity through spectral inclusions

Ancient results coming from matrix theory

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Theorem (G. Frobenius (1897))

Let $n \in \mathbb{N}$. A linear map $\Phi\colon \mathbb{M}_n \to \mathbb{M}_n$ satisfies the property

$$\det\big(\Phi(M)\big) = \det(M) \quad (M \in \mathbb{M}_n)$$

if and only if there are invertible matrices $P, Q \in \mathbb{M}_n$ with $\det(PQ) = 1$ such that either

$$\Phi(M) = PMQ \quad (M \in \mathbb{M}_n)$$

or

$$\Phi(M) = PM^tQ \quad (M \in \mathbb{M}_n),$$

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if and only if $\Phi = W\Psi$ for some automorphism or anti-automorphism Ψ of the Banach algebra $\mathbb{M}_n$ and some invertible matrix $W \in \mathbb{M}_n$ with $\det W = 1$.

Theorem (J. Dieudonné (1949))

Let $n \in \mathbb{N}$. A bijective linear map $\Phi\colon \mathbb{M}_n \to \mathbb{M}_n$ satisfies the property

$$M \in \mathbb{M}_n,\ \det(M) = 0 \ \Rightarrow\ \det\big(\Phi(M)\big) = 0$$

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Theorem (M. Marcus and R. Purves (1959))

Let $n \in \mathbb{N}$. A linear map $\Phi\colon \mathbb{M}_n \to \mathbb{M}_n$ satisfies the property

$$sp\big(\Phi(M)\big) = sp(M) \;\; (M \in \mathbb{M}_n)$$

if and only if there is an invertible matrix $P \in \mathbb{M}_n$ such that either

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Beyond finite-dimensionality

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Theorem (A. M. Gleason (1967), J. P. Kahane and W. Żelazko (1968))

Let A be a complex Banach algebra and let $\varphi\colon A \to \mathbb{C}$ be a linear functional. Then φ is multiplicative (and nonzero) if (and only if)

$$\varphi(a) \in sp(a) \quad (a \in A).$$

Kaplansky's problem

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Identify the multiplicative linear maps among all linear maps, between complex Banach algebras A and B, in terms of spectra.

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Shrinking the spectrum

$$\mathrm{sp}\big(\Phi(a)\big) \subset \mathrm{sp}(a) \quad (a \in A).$$

I. Kaplansky (1970)

Let A and B be complex Banach algebras and let $\Phi\colon A \to B$ be a linear map with the property that

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Is it true that Φ is a Jordan homomorphism, i.e.

$$\Phi(a^2) = \Phi(a)^2 \quad (a \in A)\ ?$$

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B. Aupetit (2000)

Let A and B be semisimple complex Banach algebras and let $\Phi\colon A \to B$ be a surjective linear map with the property that

$$\operatorname{sp}\big(\Phi(a)\big) = \operatorname{sp}(a) \quad (a \in A).$$

Is it true that Φ is a Jordan homomorphism?

Results in operator algebras

Theorem (A. A. Jafarian and A. R. Sourour (1986))

Let X and Y be complex Banach spaces and let $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ be a surjective linear map with the property that

$$\operatorname{sp}\big(\Phi(T)\big) = \operatorname{sp}(T) \ \ (T \in \mathcal{B}(X)).$$

Then Φ has the form $\Phi(T) = STS^{-1}$ $(T \in \mathcal{B}(X))$ for some isomorphism $S\colon X \to Y$ or $\Phi(T) = RT^*R^{-1}$ $(T \in \mathcal{B}(Y))$ for some isomorphism $R\colon X^* \to Y$.

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Theorem (A. R. Sourour (1996))

Let X and Y be Banach spaces, let A and B be unital standard operator algebras on X and Y, respectively, and let $\Phi\colon A \to B$ be a linear map. Then Φ has the form $\Phi(T) = STS^{-1}$ $(T \in A)$ for some isomorphism $S\colon X \to Y$ or $\Phi(T) = RT^{*}R^{-1}$ $(T \in A)$ for some isomorphism $R\colon X^{*} \to Y$ in either of the following cases:

1. *the map Φ is bijective and $\mathrm{sp}\big(\Phi(T)\big) \subset \mathrm{sp}(T)$ for each $T \in A$, or*
2. *the map Φ is surjective and $\mathrm{sp}\big(\Phi(T)\big) = \mathrm{sp}(T)$ for each $T \in A$.*

Linear maps preserving some parts of the spectrum

- left spectrum
- right spectrum
- approximate point spectrum
- surjectivity spectrum
- boundary of the spectrum

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- right spectrum
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Linear maps preserving some spectral quantities

- spectral radius
- essential spectral radius
- minimum modulus
- reduced minimum modulus

Part II

Introducing the problem

Approximate Gleason-Kahane-Żelazko theorem

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B. E. Johnson (1986) replaced

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Theorem (B. E. Johnson (1986))

Let A be a commutative Banach algebra. Then for each $\varepsilon > 0$ there is $\delta > 0$ such that if $\varphi\colon A \to \mathbb{C}$ is a linear functional with

$$\operatorname{dist}(\varphi(a), sp(a)) < \delta \ \ (a \in A, \ \|a\| = 1),$$

then

$$\sup\{|\varphi(ab) - \varphi(a)\varphi(b)| \colon \ a, b \in A, \ \|a\| = \|b\| = 1\} < \varepsilon.$$

AMNM algebras

A Banach algebra A is **AMNM** if for each $\varepsilon > 0$ there is $\delta > 0$ such that if φ is a linear functional on A with

$$\sup\{|\varphi(ab) - \varphi(a)\varphi(b)| : \; a, b \in A, \; \|a\| = \|b\| = 1\} < \delta,$$

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Examples

1. B. E. Johnson (1986): $C_0(\Omega)$ for each locally compact Hausdorff space Ω, $L^1(G)$ for each locally compact abelian group G, $\ell^1(\mathbb{Z}^+)$, $L^1(]0, +\infty[)$, $A(\mathbb{D})$.
2. R. A. J. Howey (2003): $C^n([0, 1])$ for each $n \in \mathbb{N}$ and certain Banach algebras of Lipschitz functions.
3. B. E. Johnson (1988): all finite-dimensional and all amenable Banach algebras. In particular, the group algebra $L^1(G)$ for each amenable group.
4. Properly infinite unital Banach algebras. In particular $\mathcal{B}(X)$ for each Banach space X which contains a complemented subspace isomorphic to $X \oplus X$.

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Part III

Celebrities from this pattern of thinking

Replacing the spectra: the pseudospectra

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ε–pseudospectrum of $a \in A$

$$\mathrm{sp}_\varepsilon(a) = \left\{ z \in \mathbb{C} \colon \left\| (a - z\mathbf{1})^{-1} \right\| > \varepsilon^{-1} \right\} \quad (\varepsilon > 0)$$

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Applications

- atmospheric science
- control theory
- ecology
- hydrodynamic stability
- lasers
- magnetohydrodynamics
- Markov chains
- non-hermitian quantum mechanics
- numerical solutions of odes/pdes
- rounding error analysis

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Johnson condition

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Our condition

$$\varphi(a) \in \mathrm{sp}_\varepsilon(a) \quad (a \in A,\ \|a\| = 1).$$

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Aupetit (preserving) condition

$$\mathsf{sp}\big(\Phi(a)\big) = \mathsf{sp}(a) \quad (a \in A).$$

Approximate preserving condition

$$\mathrm{dist}_{\mathsf{H}}\big(\mathrm{sp}(\Phi(a)), \mathrm{sp}(a)\big) < \varepsilon \quad (a \in A,\ \|a\| = 1).$$

Measuring multiplicativity

Let A and B be Banach algebras and let $\Phi\colon A \to B$ be a linear map.

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$$\mathrm{amult}(\Phi) = \sup\{\|\Phi(ab) - \Phi(b)\Phi(a)\|\colon\ a, b \in A,\ \|a\| = \|b\| = 1\}.$$

$$\Phi \text{ is a homomorphism } \Leftrightarrow\ \mathrm{mult}(\Phi) = 0$$

$$\Phi \text{ is an anti-homomorphism } \Leftrightarrow\ \mathrm{amult}(\Phi) = 0.$$

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$$\Phi \text{ is a homomorphism } \Leftrightarrow \mathrm{mult}(\Phi) = 0$$

$$\Phi \text{ is an anti-homomorphism } \Leftrightarrow \mathrm{amult}(\Phi) = 0.$$

Standard problem

Determine whether $\mathrm{mult}(\Phi)$ being small implies $\mathrm{dist}\big(\Phi, \mathrm{Hom}(A, B)\big)$ being small. Similar question for $\mathrm{amult}(\Phi)$.

Part IV

Our results

Approximate Gleason-Kahane-Żelazko theorem

Theorem

Let A be a unital Banach algebra. Then the following assertions hold.

1. *For each $\varepsilon, \nu > 0$ there is $\delta > 0$ such that if φ is a linear functional on A with*
$$\varphi(a) \in \mathrm{sp}_\delta(a) \ \ (a \in A, \|a\| = 1),$$
then $\mathrm{mult}(\varphi) < \varepsilon$ and $\|\varphi\| > \nu$.

2. *For each $\varepsilon, \nu > 0$ there is $\delta > 0$ such that if φ is a linear functional on A with $\mathrm{mult}(\varphi) < \delta$ and $\|\varphi\| > \nu$, then*
$$\varphi(a) \in \mathrm{sp}_\varepsilon(a) \ \ (a \in A, \|a\| = 1).$$

Maps approximately shrinking the spectrum

Maps approximately shrinking the spectrum

Theorem

Let X and Y be superreflexive Banach spaces. Then the following assertions hold.

1. *For each $K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ is a bijective linear map with*

$$\mathrm{sp}\big(\Phi(T)\big) \subset \mathrm{sp}_\delta(T) \;\; (T \in \mathcal{B}(X), \|T\| = 1)$$

and $\|\Phi\|, \|\Phi^{-1}\| < K$, then

$$\min\big\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\big\} < \varepsilon.$$

2. *For each $K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ is a bijective linear map with*

$$\min\big\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\big\} < \delta$$

and $\|\Phi\|, \|\Phi^{-1}\| < K$, then

$$\mathrm{sp}\big(\Phi(T)\big) \subset \mathrm{sp}_\varepsilon(T) \;\; (T \in \mathcal{B}(X), \|T\| = 1).$$

Question

Is it possible to remove the superreflexivity from the spaces X and Y in the theorem? Which conditions on the Banach spaces X and Y imply that the theorem still work?

Theorem

Let H be a separable Hilbert space. Then the following assertions hold.

1. *For each* $K, \varepsilon > 0$ *there is* $\delta > 0$ *such that if* $\Phi \colon \mathcal{B}(H) \to \mathcal{B}(H)$ *is a bijective linear map with*

$$\mathrm{sp}\big(\Phi(T)\big) \subset \mathrm{sp}_\delta(T) \ \ (T \in \mathcal{B}(H), \|T\| = 1)$$

and $\|\Phi\|, \|\Phi^{-1}\| < K$, *then*

$$\|\Phi - \Psi\| < \varepsilon$$

for some automorphism or anti-automorphism $\Psi \colon \mathcal{B}(H) \to \mathcal{B}(H)$.

2. *For each* $K, \varepsilon > 0$ *there is* $\delta > 0$ *such that if* $\Phi \colon \mathcal{B}(H) \to \mathcal{B}(H)$ *is a continuous linear map with* $\|\Phi\| < K$ *and*

$$\|\Phi - \Psi\| < \delta$$

for some automorphism or anti-automorphism $\Psi \colon \mathcal{B}(H) \to \mathcal{B}(H)$, *then*

$$\mathrm{sp}\big(\Phi(T)\big) \subset \mathrm{sp}_\varepsilon(T) \ \ (T \in \mathcal{B}(H), \|T\| = 1).$$

Question

Does the theorem still work with H replaced by a superreflexive Banach space X? Which conditions on the Banach space X imply that the theorem remain valid with H replaced by X?

Maps approximately preserving the spectrum

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Theorem

Let X and Y be superreflexive Banach spaces. Then for each $k, K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ is a surjective linear map with

$$\mathrm{dist}_{\mathrm{H}}\Big(\mathrm{sp}\big(\Phi(T)\big), \mathrm{sp}(T)\Big) < \delta \ \ (T \in \mathcal{B}(X), \|T\| = 1),$$

$\kappa(\Phi) > k$, and $\|\Phi\| < K$, then

$$\min\big\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\big\} < \varepsilon.$$

The **surjectivity modulus** of $\Phi \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is defined by

$$\kappa(T) = \sup\left\{\varrho \geq 0\colon\ \varrho\mathbb{B}_{\mathfrak{Y}} \subset \Phi\big(\mathbb{B}_{\mathfrak{X}}\big)\right\}.$$

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Is it possible to remove the superreflexivity from the spaces X and Y? Which conditions on the Banach spaces X and Y imply that the theorem still work?

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Let H be a separable Hilbert space. Then for each $k, K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{B}(H) \to \mathcal{B}(H)$ is a surjective linear map with

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Question

Does the theorem still work with H replaced by a superreflexive Banach space X? Which conditions on the Banach space X imply that the theorem remain valid with H replaced by X?

Part V

Method of proof

The main tool: ultrapowers

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and let $\mathfrak{X}$ be a Banach space. Then the **ultrapower** of $\mathfrak{X}$ with respect to $\mathcal{U}$ is the Banach space

$$\mathfrak{X}^{\mathcal{U}} = \ell^{\infty}(\mathfrak{X})/\mathfrak{N}_{\mathcal{U}},$$

where $\mathfrak{N}_{\mathcal{U}} := \{x \in \ell^{\infty}(\mathfrak{X}) : \ \lim_{\mathcal{U}} \|x_n\| = 0\}$. The norm on $\mathfrak{X}^{\mathcal{U}}$ is given by

$$\|\mathbf{x}\| = \lim_{\mathcal{U}} \|x_n\| \ \ (\mathbf{x} = (x_n) \in \mathfrak{X}^{\mathcal{U}}).$$

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There is an isometric linear map $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})^{\mathcal{U}} \to \mathcal{B}\left(\mathfrak{X}^{\mathcal{U}}, \mathfrak{Y}^{\mathcal{U}}\right)$ which is defined by

$$\mathbf{T}(\mathbf{x}) = \left(T_n(x_n)\right) \quad \left(\mathbf{T} = (T_n) \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})^{\mathcal{U}}, \mathbf{x} = (x_n) \in \mathfrak{X}^{\mathcal{U}}\right).$$

The main tool: ultrapowers

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and let $\mathfrak{X}$ be a Banach space. Then the **ultrapower** of $\mathfrak{X}$ with respect to $\mathcal{U}$ is the Banach space

$$\mathfrak{X}^{\mathcal{U}} = \ell^\infty(\mathfrak{X})/\mathfrak{N}_{\mathcal{U}},$$

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$$\mathbf{T}(\mathbf{x}) = \big(T_n(x_n)\big) \ \ \left(\mathbf{T} = (T_n) \in \mathcal{B}(\mathfrak{X},\mathfrak{Y})^{\mathcal{U}}, \mathbf{x} = (x_n) \in \mathfrak{X}^{\mathcal{U}}\right).$$

There is a canonical map $(\mathfrak{X}^*)^{\mathcal{U}} \to (\mathfrak{X}^{\mathcal{U}})^*$ given by

$$\langle \mathbf{f}, \mathbf{x}\rangle = \lim_{\mathcal{U}} \langle f_n, x_n\rangle \ \ (\mathbf{f} = (f_n) \in (\mathfrak{X}^*)^{\mathcal{U}}, \mathbf{x} = (x_n) \in \mathfrak{X}^{\mathcal{U}}).$$

This map is an isometry, and so we identify $(\mathfrak{X}^*)^{\mathcal{U}}$ with a closed subspace of $(\mathfrak{X}^{\mathcal{U}})^*$. It is known that $(\mathfrak{X}^*)^{\mathcal{U}} = (\mathfrak{X}^{\mathcal{U}})^*$ if and only if the Banach space $\mathfrak{X}$ is superreflexive.

For each $K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ is a bijective linear map with $\operatorname{sp}\big(\Phi(T)\big) \subset \operatorname{sp}_\delta(T)$ $\ (T \in \mathcal{B}(X), \|T\| = 1)$ and $\|\Phi\|, \|\Phi^{-1}\| < K$, then $\min\big\{\operatorname{mult}(\Phi), \operatorname{amult}(\Phi)\big\} < \varepsilon$.

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Suppose the assertion fails to be true. Then there exist $\tau > 0$ and a sequence (Φ_n) of bijective linear maps from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ with the properties that

$$\mathrm{sp}\big(\Phi_n(T)\big) \subset \mathrm{sp}_{1/n}(T)\ \ (T \in \mathcal{B}(X), \|T\| = 1),$$

$$\|\Phi_n\|, \|\Phi_n^{-1}\| < K,$$

$$\mathrm{mult}(\Phi_n), \mathrm{amult}(\Phi_n) \geq \tau.$$

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Then the map $\Phi = (\Phi_n)\colon \mathcal{B}(X)^{\mathcal{U}} \subset \mathcal{B}(X^{\mathcal{U}}) \to \mathcal{B}(Y)^{\mathcal{U}} \subset \mathcal{B}(Y^{\mathcal{U}})$ is a continuous bijective linear map.

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$$\mathrm{sp}\big(\Phi(\mathbf{T})\big) \subset \mathrm{sp}(\mathbf{T}) \ \ (\mathbf{T} = (T_n) \in \mathcal{B}(X)^{\mathcal{U}}).$$

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$$\sup_{\|T\|} \mathrm{dist}_{\mathrm{H}}\Big(\mathrm{sp}\big(\Phi(T)\big), \mathrm{sp}(T)\Big) \to 0,$$

$$\kappa(\Phi_n) > k, \;\; \|\Phi_n\| < K,$$

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Then the map $\Phi = (\Phi_n)\colon \mathcal{B}(X)^{\mathcal{U}} \subset \mathcal{B}\big(X^{\mathcal{U}}\big) \to \mathcal{B}(Y)^{\mathcal{U}} \subset \mathcal{B}\big(Y^{\mathcal{U}}\big)$ is a continuous surjective linear map.

$$\mathrm{sp}\big(\Phi(\mathbf{T})\big) = \mathrm{sp}(\mathbf{T}) \;\; (\mathbf{T} = (T_n) \in \mathcal{B}(X)^{\mathcal{U}}).$$

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Let A and B be Banach algebras and let $\Phi \in \mathcal{B}(A, B)$ such that $\|\Phi - \Psi\| < \varepsilon$ for some continuous homomorphism or anti-homomorphism $\Psi\colon A \to B$. Then it is straightforward to check that

$$\min\big\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\big\} \leq (1 + \varepsilon + 2\|\Phi\|)\varepsilon.$$

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The **AMNM** problem is concerned with the question of whether the constant $\min\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\}$ being small implies Φ is near an homomorphism or anti-homomorphism $\Psi\colon A \to B$:

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for each $K, \varepsilon > 0$ is there $\delta > 0$ such that if $\Phi \in \mathcal{B}(A, B)$ with $\|\Phi\| < K$ and $\min\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\} < \delta$ then $\|\Phi - \Psi\| < \varepsilon$ for some continuous homomorphism or anti-homomorphism $\Psi\colon A \to B$?

Examples

The answer is afirmative in any of the following cases.

1. The Banach algebras A and B are finite-dimensional.
2. The Banach algebra A is finite-dimensional and semisimple and B is any Banach algebra.
3. The Banach algebra A is amenable and B is a two-sided ideal of a dual Banach algebra C in the sense that there is a Banach C-bimodule C_* so that C is isomorphic as a C-bimodule with $(C_*)^*$. As a matter of fact, this applies to the pairs:
   - $(L^1(G_1), M(G_2))$ and $(L^1(G_1), L^1(G_2))$ for each amenable group G_1 and each locally compact group G_2,
   - $(\mathcal{K}(H_1), \mathcal{B}(H_2))$ and $(\mathcal{K}(H_1), \mathcal{K}(H_2))$ for all Hilbert spaces H_1 and H_2.
4. $(\mathcal{B}(H), \mathcal{B}(H))$ for each separable Hilbert space H.

For each $K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ is a bijective linear map with $\min\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\} < \delta$ and $\|\Phi\|, \|\Phi^{-1}\| < K$, then $\mathrm{sp}(\Phi(T)) \subset \mathrm{sp}_\varepsilon(T)$ $(T \in \mathcal{B}(X), \|T\| = 1)$.

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Suppose the assertion is false. Then there exist $\tau > 0$, a sequence (Φ_n) of bijective linear maps from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$, and (T_n) in $\mathcal{B}(X)$ such that

$$\lim \min\{\mathrm{mult}(\Phi_n), \mathrm{amult}(\Phi_n)\} = 0,$$

$$\|\Phi_n\|, \|\Phi_n^{-1}\| < K,$$

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We now consider the bijective continuous linear map

$$\Phi = (\Phi_n) \colon \mathcal{B}(X)^{\mathcal{U}} \subset \mathcal{B}\big(X^{\mathcal{U}}\big) \to \mathcal{B}(Y)^{\mathcal{U}} \subset \mathcal{B}\big(Y^{\mathcal{U}}\big).$$

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Φ is either an isomorphism or an anti-isomorphism.

For each $K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ is a bijective linear map with $\min\{\operatorname{mult}(\Phi), \operatorname{amult}(\Phi)\} < \delta$ and $\|\Phi\|, \|\Phi^{-1}\| < K$, then $\operatorname{sp}(\Phi(T)) \subset \operatorname{sp}_\varepsilon(T) \ \ (T \in \mathcal{B}(X), \|T\| = 1)$.

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Φ is either an isomorphism or an anti-isomorphism. Hence $\operatorname{sp}(\Phi(\mathbf{T})) = \operatorname{sp}(\mathbf{T})$ and this implies that for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ with $\operatorname{sp}(\Phi_n(T_n)) \subset \operatorname{sp}_\varepsilon(T_n)$.

Part VI

Concluding remarks

Theorem

Let $\mathcal{A}_1(X)$ and $\mathcal{A}_2(Y)$ be unital standard operator algebras on superreflexive Banach spaces Banach spaces X and Y. Then the following assertions hold.

1. *For each $K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{A}_1(X) \to \mathcal{A}_2(Y)$ is a bijective linear map with*

$$\mathrm{sp}\big(\Phi(T)\big) \subset \mathrm{sp}_\delta(T) \;\; (T \in \mathcal{A}_1(X), \|T\| = 1)$$

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Theorem

Let $\mathcal{A}_1(X)$ and $\mathcal{A}_2(Y)$ be unital standard operator algebras on superreflexive Banach spaces X and Y. Then for each $k, K, \varepsilon > 0$ there is $\delta > 0$ such that if $\Phi\colon \mathcal{A}_1(X) \to \mathcal{A}_2(Y)$ is a surjective linear map with

$$\mathrm{dist_H}\Big(\mathrm{sp}\big(\Phi(T)\big), \mathrm{sp}(T)\Big) < \delta \ \ (T \in \mathcal{A}_1(X), \|T\| = 1),$$

$\kappa(\Phi) > k$, and $\|\Phi\| < K$, then

$$\min\big\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\big\} < \varepsilon.$$

Question

Let $\mathcal{A}_1(H)$ and $\mathcal{A}_2(H)$ be unital standard operator algebras on a separable Hilbert space H. Is the pair $(\mathcal{A}_1(H), \mathcal{A}_2(H))$ AMNM?

Quantitative estimates

Quantitative estimates

Approximate Gleason-Kahane-Żelazko theorem

1. $\varphi(a) \in \mathrm{sp}_\varepsilon(a) \ \ (a \in A, \|a\| = 1) \ \ \Rightarrow$
   $\mathrm{mult}(\varphi) \le \frac{4(1+\varepsilon)}{\log(1/\varepsilon)} \left(1 + \frac{18}{(\log 2)^2}\right) + \frac{\varepsilon(1+\varepsilon)^2}{1-\varepsilon}.$
2. Either $\|\varphi\| \le \frac{\mathrm{mult}(\varphi)}{1-\mathrm{mult}(\varphi)^{1/2}}$ or $\varphi(a) \in \mathrm{sp}_\varepsilon(a) \ \ (a \in A, \|a\| = 1)$ with
   $$\varepsilon = \frac{2\mathrm{mult}(\varphi)\left(3+\sqrt{1+4\mathrm{mult}(\varphi)}\right)+\mathrm{mult}(\varphi)^{1/2}\left(3+\sqrt{1+4\mathrm{mult}(\varphi)}\right)^2}{4\left(1-\mathrm{mult}(\varphi)^{1/2}\right)}.$$

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Maps approximately shrinking/preserving the spectrum

?