Boundedness of Toeplitz operators on Bergman spaces

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Introduction

We report on recent results concerning boundedness of Toeplitz operators on Bergman spaces with $1 < p < \infty$. The results extend to compactness and Fredholm theory, also in the case $p = 1$, and to the setting of Fock and Dirichlet spaces.

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Introduction

Notation: $\mathbb{D} = \{|z| < 1\}$, dA normalized area measure on $\mathbb{D}$. Consider the space $L^p := (L^p(\mathbb{D}, dA), \| \cdot \|_p)$ and the *Bergman space* A^p, which is the closed subspace of L^p consisting of analytic functions. The *Bergman projection* P is the orthogonal projection of L^2 onto A^2:

$$Pf(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta) \ , \ f \in L^2, \ z \in \mathbb{D}.$$

Known to be a bounded projection of L^p onto A^p, when $1 < p < \infty$. For an integrable function $a : \mathbb{D} \to \mathbb{C}$ and, say, bounded analytic functions f, define the *Toeplitz operator* T_a *with symbol* a by setting

$$T_a f = P(af).$$

Since P is bounded, it follows easily that T_a extends to a bounded operator $A^p \to A^p$ for $1 < p < \infty$, whenever a is a bounded measurable function. A considerably more difficult question is the characterization of boundedness of T_a on A^p for general unbounded symbols = *well known open problem* even for $p = 2$.

Locally integrable symbols

Denote by $\mathcal{D}$ the family of sets $D := D(r, \theta)$ (similar to hyperbolic discs)

$$D = \{\rho e^{i\phi} \mid r \leq \rho \leq 1 - \frac{1}{2}(1-r) \ , \ \theta \leq \phi \leq \theta + \pi(1-r)\}$$

for all $0 < r < 1$, $\theta \in [0, 2\pi]$. Not difficult to show: a sufficient condition for the boundedness of $T_a : A^p \to A^p$ is

$$\sup_{D \in \mathcal{D}} |D|^{-1} \int_D |a(z)| dA(z) < \infty$$

Given $D = D(r, \theta) \in \mathcal{D}$ and $\zeta = \rho e^{i\phi} \in D$, denote

$$\hat{a}_D(\zeta) := \frac{1}{|D|} \int_r^\rho \int_\theta^\phi a(\varrho e^{i\varphi}) \varrho d\varphi d\varrho.$$

Theorem (Paper 2)

The operator $T_a : A^p \to A^p$ is well defined and bounded for all $1 < p < \infty$, if there exists a constant $C > 0$ such that $|\hat{a}_D(\zeta)| \leq C$ for all $D \in \mathcal{D}$, for all $\zeta \in D$.

Radial symbols

Next assume that the symbol $a \in L^1(\mathbb{D})$ is radial: $a(z) = a(|z|)$.
For all $n \in \mathbb{N}$, $n \geq 1$, we define the n:th indefinite integrals by

$$I_a(r) := I_a^{(1)}(r) = \int_r^1 a(s)sds, \;\; I_a^{(n+1)}(r) := \int_r^1 I_a^{(n)}(s)sds, \;\; r \in [0,1[.$$

It is clear that for positive a all indefinite integrals $I_a^{(n)}$ are positive functions; moreover, if $I_a^{(n)}$ is positive, then the same is true for all $I_a^{(k)}$ with $k > n$. On the other hand, $I_a^{(n)}$ may very well be positive, though a is not.

Theorem (Paper 6)

Let $1 < p < \infty$. Assume that for some $n \geq 1$ the function I_a^n is nonnegative. Then the Toeplitz operator $T_a : A^p \to A^p$ is bounded, if and only if

$$|I_a^{(n+1)}(r)| \leq C(1-r)^{n+1}$$

for all $r \in [0,1[$.

Radial symbols

Relation to coefficient multipliers on the Hardy space $H^p = H^p(\mathbb{D})$. For all k, define

$$\gamma_k = 2\int_0^1 (k+1) r^{2k+1} a(r) dr.$$

The following result holds in Bergman spaces with radial weighted norms $\|f\|_{p,\mu}^p := (2\pi)^{-1} \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p r d\theta d\mu(r)$
where μ is a quite general bounded positive measure on $[0,1[$.

Theorem (Paper 6)

$T_a : A_\mu^p \to A_\mu^p$ is bounded, if and only if the multipliers

$$T_a^{(n)} : \sum_{k\in\mathbb{N}} f_k e^{ik\theta} \mapsto \sum_{k\in\mathbb{N}_n} \gamma_k f_k e^{ik\theta} \quad , \ \theta \in [0, 2\pi],$$

are for all $n \in \mathbb{N}$ uniformly bounded operators $H^p \to H^p$. Moreover, $T_a : A_\mu^p \to A_\mu^p$ is compact, if and only if the sequence formed by the operator norms of $T_a^{(n)} : H^p \to H^p$ converges to 0.

The sets $\mathbb{N}_n \subset \mathbb{N}$ are disjoint and finite, and $\cup_n \mathbb{N}_n = \mathbb{N}$.

Distributional symbols

Using the familiar dual pairing for distributions on $\mathbb{D} \subset \mathbb{R}^2$ we can write

$$T_a f(z) = \int_{\mathbb{D}} \frac{a(\zeta) f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta) = \langle f(\zeta)(1 - z\bar{\zeta})^{-2}, a \rangle_\zeta ,$$

if for example $a \in L^1(\mathbb{D})$. The last expression on the right hand side is also defined for all *compactly supported distributions on* $\mathbb{D}$.
Define the weight function $\nu : \mathbb{D} \to \mathbb{R}^+$ by $\nu(z) = 1 - |z|^2$. Given $m \in \mathbb{N}$, denote by $W_\nu^{m,1}(\mathbb{D})$ the weighted Sobolev space of functions f on $\mathbb{D}$ with

$$\| f ; W_\nu^{m,1} \| := \sum_{|\alpha| \le m} \int_{\mathbb{D}} |D^\alpha f(z)| \nu(z)^{|\alpha|} dA(z) < \infty .$$

Its dual space is $W_\nu^{-m,\infty}(\mathbb{D})$ the weighted Sobolev space of distributions a on $\mathbb{D}$ which can be written as

$$a = \sum_{0 \le |\alpha| \le m} (-1)^{|\alpha|} D^\alpha b_\alpha ,$$

where $\operatorname{ess\,sup}_{z \in \mathbb{D}} \nu(z)^{-|\alpha|} |b_\alpha(z)| =: \| b_\alpha ; L^\infty_{\nu^{-|\alpha|}} \| < \infty$ for all α. Norm of a is $\| a ; W_\nu^{-m,\infty} \| := \inf \max_{|\alpha| \le m} \| b_\alpha ; L^\infty_{\nu^{-|\alpha|}} \|$.

Distributional symbols

Theorem (Paper 3)

Assume that the distribution a belongs to $W_\nu^{-m,\infty}$ for some m. Then the Toeplitz operator T_a, defined by the formula

$$T_a f(z) = \sum_{0 \le |\alpha| \le m} \int_{\mathbb{D}} \Big(D_\zeta^\alpha \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} \Big) b_\alpha(\zeta) dA(\zeta) \ , \ f \in A^p,$$

is well defined and bounded $A^p \to A^p$ for all $1 < p < \infty$. The resulting operator is independent of the choice of the representation of a. Moreover, there is a constant $C > 0$ such that

$$\| T_a : A^p \to A^p \| \le C \| a; W_\nu^{-m,\infty} \|.$$

Remark. This theorem makes also sense for symbols which are locally integrable functions. Actually we are lead to the hypothesis that the sufficient condition in this theorem [3] is weaker than in the one for locally integrable symbols [2]. In the paper [5] this hypothesis was proven for radial symbols.