Bounded sets in topological groups

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The character of topological groups via Pontryagin duality

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Joint work with C. Chis, M.V. Ferrer and B. Tsaban

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1. 1. Introduction

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Let X be a Thychonoff space. The *free abelian topological group* $A(X)$ over X is defined as the abelian topological group such that each continuous function from X into any abelian topological group H has a unique extension to a continuous homomorphism $\varphi : A(X) \longrightarrow H$.

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Theorem [Nickolas and Tkachenko]

Let X be a compact space. Then

$$\chi(A(X)) = \mathfrak{d} \cdot \mathrm{cof}([\mathrm{w}(\mathrm{X})]^{\omega}).$$

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Let X be a compact space. Then

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The proof of this result uses elementary but very sophisticated topological arguments. In the sequel, we are going to look at this question having in mind Pontryagin duality.

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$$\langle\, ,\rangle : \widehat{G} \times G \longrightarrow \mathbb{T}\,,$$

and a canonical evaluation homomorphism

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Theorem [Pontryagin-van Kampen]

The evaluation homomorphism $\mathcal{E}_G$ is an isomorphism of topological groups.

The group G is compact if and only if the dual group $\widehat{G}$ is discrete.

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We have

$$\widehat{\mathbb{T}} \cong \mathbb{Z},\ \widehat{\mathbb{Z}} \cong \mathbb{T},\ \widehat{\mathbb{R}} \cong \mathbb{R},\ \widehat{(\mathbb{Z}/n)} \cong \mathbb{Z}/n.$$

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Pontryagin duality goes hand-in-hand with the theory of Fourier transform which I briefly describe here.
There is a positive measure dx on G called the Haar measure. It is the unique (up to scalar multiple) Borel measure which is invariant under translations.

Using it, we can define the spaces $L^1(G)$ and $L^2(G)$ of integrable and square integrable functions.

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The *Fourier transform*

$$\mathcal{F} : L^1(G) \cap L^2(G) \longrightarrow L^2(\widehat{G})$$

is defined as follows

$$(\mathcal{F}f)(\chi) = \int_G f(x)\overline{\langle \chi, x \rangle} dx.$$

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The set $L^1(G) \cap L^2(G)$ is a dense subspace of $L^2(G)$ and we have:

Theorem [Plancherel]

With correct normalization of the Haar measure $d\chi$ on $\widehat{G}$, the mapping $f \to \mathcal{F}f$ extends uniquely to an isometry

$$\mathcal{F} : L^2(G) \longrightarrow L^2(\widehat{G}).$$

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If G denotes a compact abelian group and Γ denotes its dual group, we have the following equivalences between topological properties of G and algebraic properties of Γ:

(i) $weight(G) = |\Gamma|$ (metrizablity$\Leftrightarrow |\Gamma| \leq \omega$);
(ii) G is connected $\Leftrightarrow \Gamma$ is torsion free;
(iii) $Dim(G) = 0 \Leftrightarrow \Gamma$ is torsion; and
(iv) G is monothetic $\Leftrightarrow \Gamma$ is isomorphic to a subgroup of $\mathbb{T}_d$.

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If the evaluation map is just an embedding, we say that G is *subreflexive*. In this case, we identify G with its image $\mathcal{E}[G] \leq \widehat{\widehat{G}}$.

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We say that a topological abelian group (G, τ) satisfies *Pontryagin duality* (*P-reflexive* for short) if the evaluation map $\mathcal{E}_G$ is a topological isomorphism onto.
If the evaluation map is just an embedding, we say that G is *subreflexive*. In this case, we identify G with its image $\mathcal{E}[G] \leq \widehat{\widehat{G}}$.
Let $\mathcal{K}(G)$ denote the family of all compact subsets of G. For a set $A \subseteq G$ and a positive real ϵ, define (additive notation)

$$[A, \epsilon] = \{\chi \in \widehat{G} : (\forall a \in A)\ |\chi(a)| \leq \epsilon\}.$$

The sets $[K, \epsilon] \subseteq \widehat{G}$ $(K \in \mathcal{K}(G))$ form a neighborhood base at the trivial character in $\widehat{G}$.

Observe that for each compact subset K of G, the set $K_n = K \cup 2K \cup \cdots \cup nK$ is also compact, and $[K_n, 1/4] \subset [K, 1/4n]$.

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Definition

For $A \subseteq G$, $A^{\triangleright} = [A, 1/4]$
for $X \subseteq \widehat{G}$, $X^{\triangleleft} = \{g \in G : (\forall \chi \in X) \ |\chi(g)| \leq \frac{1}{4}\}$.

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Definition [Vilenkin]

A set $A \subseteq G$ is *quasi-convex* if $A^{\triangleright\triangleleft} = A$. G is *locally quasi-convex* if it has a neighborhood base at its identity consisting of quasi-convex sets.

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Lemma [Banaszczyk]

For each neighborhood U of 0 in G, $U^{\triangleright} \in \mathcal{K}(\widehat{G})$.

In a different direction, we need the following definitions.

Definition

Let $(\mathbb{P}, \leq)$ be a partial ordered set. A subset D is **cofinal** in $\mathbb{P}$ if $D \subset \mathbb{P}$, and for each $p \in \mathbb{P}$ there is $d \in D$ such that $p \leq d$. The **cofinality** of $\mathbb{P}$, denoted $\mathrm{cof}(\mathbb{P})$, is the minimal cardinality of a cofinal set in $\mathbb{P}$.

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Given two partial ordered sets $(X, \leq)$ and $(Y, \tilde{\leq})$ we say that $(X, \leq)$ has **cofinality greater or equal than** $(Y, \tilde{\leq})$ when there is a map $\Phi : X \to Y$ which preserves the order and such that $\Phi(X)$ is cofinal in Y. **Cofinality equivalence** is defined accordingly.

For each $A \subseteq G$, $A^{\triangleright}$ is always a quasi-convex subset of $\widehat{G}$. Thus, $\widehat{G}$ is locally quasi-convex for all topological groups G. Moreover, local quasi-convexity is hereditary for arbitrary subgroups.

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Lemma

Let G be a complete locally quasi-convex group. Let $\mathcal{N}_{\widehat{G}}$ be the family of all neighborhoods of 0 in $\widehat{G}$. Then:

- $(\mathcal{N}_{\widehat{G}}, \supseteq)$ is cofinally equivalent to $(\mathcal{K}(G), \subseteq)$.
- $\chi(\widehat{G}) = \mathrm{cof}(\mathcal{K}(\mathrm{G}))$.

[Proof]

Clearly, the polar map $\triangleright : \mathcal{K}(G) \to \mathcal{N}_{\widehat{G}}$ is monotone and cofinal. Consider the other direction. Let $K \in \mathcal{K}(G)$, and take $U = K^{\triangleright} \in \mathcal{N}_{\widehat{G}}$. By a Banaszczyk Lemma $U^{\triangleright} \in \mathcal{K}(\widehat{\widehat{G}})$. Now,

$$K \subset K^{\triangleright\triangleleft} = U^{\triangleleft} = \mathcal{E}^{-1}[U^{\triangleright} \cap \mathcal{E}[G]].$$

Clearly $U^{\triangleright} \cap \mathcal{E}[G]$ is precompact and, since $\mathcal{E}$ is open and G is complete, it follows that $\mathcal{E}^{-1}[U^{\triangleright} \cap \mathcal{E}[G]]$ is compact. Thus, the monotone map $\triangleleft : \mathcal{N}_{\widehat{G}} \to \mathcal{K}(G)$ is also cofinal.

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Proposition

Let G be subreflexive, and $\mathcal{N}_G$ be the family of all neighborhoods of 0 in G. Then:

1. $(\mathcal{N}_G, \supset)$ is cofinally equivalent to $(\mathcal{K}(\widehat{G}), \subset)$.
2. $\chi(G) = \mathrm{cof}(\mathcal{K}(\widehat{G}))$.

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Thus duality has taken us to the question of estimating $\mathrm{cof}(\mathcal{K}(\mathrm{G}))$ for topological abelian groups G.

Assume that G is a metrizable topological group, $\mathfrak{V} = \{V_m\}_{m<\omega}$ is a neighborhood base of the identity e, I is a dense subset of G having minimal cardinality, and $\mathcal{B}$ denotes the collection of all precompact (or *bounded*) subsets of G.

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If $[I]^*$ designates the collection of (non empty) finite subsets of I and ${}^{\omega}[I]^*$ the set of all functions $f : \omega \to [I]^*$,then we have defined a map

$$\Psi_{\mathfrak{V}} : \ {}^{\omega}[I]^* \to \mathcal{B}$$

as follows

$$\Psi_{\mathfrak{V}}(\alpha) = \bigcap_{m\in\omega} \left(\bigcup_{i\in\alpha(m)} (iV_m) \right).$$

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$$\Psi_{\mathfrak{V}}(\alpha) = \bigcap_{m\in\omega} \left(\bigcup_{i\in\alpha(m)} (iV_m) \right).$$

If we consider the inclusion order in $\mathcal{B}$ and the pointwise inclusion order " $\subseteq$ " in ${}^{\omega}[I]^*$; that is, for $\alpha, \beta \in {}^{\omega}[I]^*$, $\alpha \subseteq \beta$ means: $\alpha(n) \subseteq \beta(n)$ for all $n < \omega$.

Lemma

The map $\Psi_{\mathfrak{Y}}$ is order preserving and $\Psi_{\mathfrak{Y}}({}^{\omega}[I]^{*})$ is a cofinal subset of $\mathcal{B}$.

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Proof

It is clear that $\Psi_{\mathfrak{Y}}$ preserves the order. Now, let $B \in \mathcal{B}$. Then, for all $m \in \omega$, there is $F_m \in [I]^{*}$ such that $B \subseteq \bigcup_{i \in F_m} iV_m$. Take $\alpha : \omega \longrightarrow [I]^{*}$ defined by $\alpha(m) = F_m$. Then $\Psi_{\mathfrak{Y}}(\alpha) \supseteq B$.

Lemma

The map $\Psi_{\mathfrak{V}}$ is order preserving and $\Psi_{\mathfrak{V}}({}^{\omega}[I]^*)$ is a cofinal subset of $\mathcal{B}$.

Proof

It is clear that $\Psi_{\mathfrak{V}}$ preserves the order. Now, let $B \in \mathcal{B}$. Then, for all $m \in \omega$, there is $F_m \in [I]^*$ such that $B \subseteq \bigcup_{i \in F_m} iV_m$. Take $\alpha : \omega \longrightarrow [I]^*$ defined by $\alpha(m) = F_m$. Then $\Psi_{\mathfrak{V}}(\alpha) \supseteq B$.

Notice that for a bounded neighbourhood U of the identity in G, the set

$$\mathcal{B}_U = \left\{ \bigcup_{i \in F} iU \ : \ F \in [I]^* \right\}$$

is cofinal in $\mathcal{B}$. As a consequence, if G is locally bounded and $\kappa = |I|$, then $\mathcal{B}$ is cofinally equivalent to $[\kappa]^*$ and $\operatorname{cof}(\mathcal{B}) = \kappa$.

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$$\Theta(\alpha)(n) = max\{j : j \in \alpha(n)\}$$

establishes the cofinal equivalence between the sets ${}^{\omega}[\omega]^{*}$ and ${}^{\omega}\omega$, the later equipped with the canonical pointwise order.
Hence, we shall take the set ${}^{\omega}\omega$ (not ${}^{\omega}[\omega]^{*}$) for the sake of simplicity.

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Hence, we shall take the set ${}^{\omega}\omega$ (not ${}^{\omega}[\omega]^{*}$) for the sake of simplicity.
When $|I| = \omega$, in addition to the already defined map $\Psi_{\mathfrak{V}}$, it is also useful to consider the map

$$\Phi_{\mathfrak{V}} : \mathcal{B} \to {}^{\omega}\omega$$

defined by the rule

$$\Phi_{\mathfrak{V}}(K)(m) := min\left\{n \,:\, K \subseteq \bigcup_{i \leq n} iV_m\right\}.$$

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$\Phi_{\mathfrak{V}}$ is order preserving and relates the confinality of $\mathcal{B}$ and ${}^{\omega}\omega$.

Lemma

If G is metrizable and not locally bounded, then there is a neighborhood base $\mathfrak{V} = \{U_m\}_{m<\omega}$ such that $\Phi_{\mathfrak{V}}(\mathcal{B})$ is cofinal in ${}^{\omega}\omega$.

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It suffices to show that there is a neighborhood base $\{U_n : n \in \omega\}$, at e, such that for each $f \in {}^{\omega}\omega$, there is a compact $K \subseteq G$, with $f \leq \Phi_{\mathfrak{V}}[K]$.

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Let U_n, $n \in \mathbb{N}$, be a descending neighborhood base at e consisting of unbounded subsets. As each U_n is not bounded, we may assume (by shrinking U_{n+1} if needed) that there is no m such that $U_n \subseteq \{1, \ldots, m\} \cdot U_{n+1}$.

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Theorem

Let G be a separable metrizable group. Then $\mathcal{B}$ is cofinally equivalent to one of following ordered sets, $\{0\}$, ω, or ${}^{\omega}\omega$, depending on whether G is either trivial or bounded, locally bounded, or non locally bounded respectively.

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Let G be a separable metrizable group. Then $\mathcal{B}$ is cofinally equivalent to one of following ordered sets, $\{0\}$, ω, or ${}^{\omega}\omega$, depending on whether G is either trivial or bounded, locally bounded, or non locally bounded respectively.
Hence, if the group G is not locally bounded, it follows that $\mathrm{cof}(\mathcal{B}) = \mathfrak{d}$.

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The extension of these results to arbitrary metrizable groups G deals not only with the density $\mathrm{d(G)}$ of the group but with another invariant cardinal of the group, named *local density* of G, which is defined by

$$\mathrm{ld(G)} = \min\{\mathrm{d(U)} : \mathrm{U} \text{ is a nbd of } \mathrm{e} \text{ in } \mathrm{G}\}.$$

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Theorem

Let G be a metrizable group that is not locally bounded. Then

$$\mathrm{cof}(\mathcal{B}) = \mathfrak{d} \cdot \mathrm{d(G)} \cdot \mathrm{cof}({}^{\omega}[\mathrm{ld(G)}]).$$

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Using duality we obtain

Theorem

Let G be a subreflexive group, such that the group $\Gamma = \widehat{G}$ is metrizable. Then $\chi(G) = \mathrm{cof}(\mathcal{B}(\Gamma))$. Thus,

- If Γ is precompact, then $\chi(G) = 1$, that is, G is discrete.
- If Γ is nonprecompact locally precompact, then $\chi(G) = \mathrm{d}(\Gamma)$.
- If Γ is not locally precompact, then $\chi(G) = \mathfrak{d} \cdot \mathrm{d}(\Gamma) \cdot \mathrm{cof}([^{\omega}\mathrm{ld}(\Gamma)])$.

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- If Γ is not locally precompact, then $\chi(G) = \mathfrak{d} \cdot \mathrm{d}(\Gamma) \cdot \mathrm{cof}([^{\omega}\mathrm{ld}(\Gamma)])$.

In particular, we deduce for free abelian topological groups

Corollary

Assume that X is a k_ω space. Then

$$\chi(A(X)) = \mathfrak{d} \cdot \mathrm{cof}(^{\omega}[\kappa]),$$

where $\kappa = \sup\{w(K) : K \in \mathcal{K}(X)\}$.

Theorem

Let G be a locally quasi-convex k_ω group with $\{K_n : n \in \omega\}$ cofinal in $\mathcal{K}(G)$. Let

$$\begin{aligned} \kappa &= \sup_{n\in\mathbb{N}} w(K_n); \text{ and} \\ \lambda &= \min\{\sup\{w(K) : K \in \mathcal{K}(G/H)\} : \\ & \quad H \leq G \text{ compact}\}. \end{aligned}$$

- If G is nondiscrete and locally compact, then $\chi(G) = \kappa$.
- If G is not locally compact, then $\chi(G)$ is the maximum of $\mathfrak{d}$, κ, and $\mathrm{cof}({}^\omega[\lambda])$.