Bounded sets in topological groups

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The character of topological groups via Pontryagin duality

Joint work with C. Chis, M.V. Ferrer and B. Tsaban
1. Introduction

2. Duality of abelian groups

3. Bounded sets in topological groups

4. Main results
1. Introduction

2. Duality of abelian groups

Index
Index

1. Introduction

2. Duality of abelian groups

3. Bounded sets in topological groups
Index

1. Introduction
2. Duality of abelian groups
3. Bounded sets in topological groups
4. Main results
1. Introduction

2. Duality of abelian groups

3. Bounded sets in topological groups

4. Main results
Let $X$ be a Tychonoff space. The *free abelian topological group* $A(X)$ over $X$ is defined as the abelian topological group such that each continuous function from $X$ into any abelian topological group $H$ has a unique extension to a continuous homomorphism $\varphi : A(X) \longrightarrow H$. 
Let $X$ be a Tychonoff space. The **free abelian topological group** $A(X)$ over $X$ is defined as the abelian topological group such that each continuous function from $X$ into any abelian topological group $H$ has a unique extension to a continuous homomorphism $\varphi : A(X) \rightarrow H$.

**Theorem [Nickolas and Tkachenko]**

Let $X$ be a compact space. Then

$$\chi(A(X)) = \delta \cdot \text{cof}([w(X)]^\omega).$$
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**Theorem [Nickolas and Tkachenko]**

Let $X$ be a compact space. Then

$$\chi(A(X)) = \mathfrak{d} \cdot \text{cof}([\text{w}(X)]^\omega).$$

The proof of this result uses elementary but very sophisticated topological arguments. In the sequel, we are going to look at this question having in mind Pontryagin duality.
Let $G$ be a commutative locally-compact group. A character $\chi$ of $G$ is a continuous homomorphism $\chi : G \to \mathbb{T}$ where $\mathbb{T}$ is the multiplicative group of complex numbers of modulus 1.
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The characters form a group $\hat{G}$, named dual group, which is given the topology of uniform convergence on compact subsets of $G$. 

Theorem [Pontryagin-van Kampen]
The evaluation homomorphism $\mathcal{E}_G$ is an isomorphism of topological groups.
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$$\langle , \rangle : \hat{G} \times G \to \mathbb{T},$$

and a canonical evaluation homomorphism

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The group $G$ is compact if and only if the dual group $\hat{G}$ is discrete.
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We have

$$\hat{T} \cong \mathbb{Z}, \quad \hat{\mathbb{Z}} \cong T, \quad \hat{\mathbb{R}} \cong \mathbb{R}, \quad \widehat{\mathbb{Z}/n} \cong \mathbb{Z}/n.$$  

(some groups are self dual, such as finite abelian groups or the additive group of the real numbers).
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(some groups are self dual, such as finite abelian groups or the additive group of the real numbers).

Pontryagin duality goes hand-in-hand with the theory of Fourier transform which I briefly describe here.

There is a positive measure $dx$ on $G$ called the Haar measure. It is the unique (up to scalar multiple) Borel measure which is invariant under translations.
Using it, we can define the spaces $L^1(G)$ and $L^2(G)$ of integrable and square integrable functions.
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$$\mathcal{F} : L^1(G) \cap L^2(G) \longrightarrow L^2(\hat{G})$$

is defined as follows

$$(\mathcal{F}f)(\chi) = \int_{G} f(x) \langle \chi, x \rangle \, dx.$$
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The set $L^1(G) \cap L^2(G)$ is a dense subspace of $L^2(G)$ and we have:

**Theorem [Plancherel]**

With correct normalization of the Haar measure $d\chi$ on $\hat{G}$, the mapping $f \to \mathcal{F} f$ extends uniquely to an isometry

$$\mathcal{F} : L^2(G) \to L^2(\hat{G}).$$
1. Introduction

2. Duality of abelian groups

3. Bounded sets in topological groups

4. Main results
Pontryagin-van Kampen duality establishes a duality between the subcategories of compact and discrete abelian groups.
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Pontryagin-van Kampen duality establishes a duality between the subcategories of compact and discrete abelian groups. If $G$ denotes a compact abelian group and $\Gamma$ denotes its dual group, we have the following equivalences between topological properties of $G$ and algebraic properties of $\Gamma$:

(i) $\text{weight}(G) = |\Gamma|$ (metrizability $\iff |\Gamma| \leq \omega$);
(ii) $G$ is connected $\iff \Gamma$ is torsion free;
(iii) $\text{Dim}(G) = 0$ $\iff \Gamma$ is torsion; and
(iv) $G$ is monothetic $\iff \Gamma$ is isomorphic to a subgroup of $\mathbb{T}_d$. 

Salvador Hernández

Bounded sets in topological groups
The notions defined for LCA groups previously (characters, dual group,...) make sense for general topological abelian groups.
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Let \(\mathcal{K}(G)\) denote the family of all compact subsets of \(G\). For a set \(A \subseteq G\) and a positive real \(\epsilon\), define (additive notation)

\[
[A, \epsilon] = \{\chi \in \widehat{G} : (\forall a \in A) \ |\chi(a)| \leq \epsilon\}.
\]

The sets \([K, \epsilon] \subseteq \widehat{G} (K \in \mathcal{K}(G))\) form a neighborhood base at the trivial character in \(\widehat{G}\).
Observe that for each compact subset $K$ of $G$, the set $K_n = K \cup 2K \cup \cdots \cup nK$ is also compact, and $[K_n, 1/4] \subset [K, 1/4n]$. 
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is also compact, and

$$[K_n, 1/4] \subset [K, 1/4n].$$

Thus, the sets $[K, 1/4], K \in \mathcal{K}(G)$, also form a neighborhood base of $\hat{G}$ at the trivial character.

**Definition**

For $A \subseteq G$, $A^\triangleright = [A, 1/4]$

for $X \subseteq \hat{G}$, $X^\triangleleft = \{g \in G : (\forall \chi \in X) \ |\chi(g)| \leq \frac{1}{4}\}$. 
Observe that for each compact subset $K$ of $G$, the set $K_n = K \cup 2K \cup \cdots \cup nK$ is also compact, and $[K_n, 1/4] \subset [K, 1/4n]$. Thus, the sets $[K, 1/4], K \in \mathcal{K}(G)$, also form a neighborhood base of $\hat{G}$ at the trivial character.

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For $A \subseteq G$, $A^\triangleright = [A, 1/4]$ for $X \subseteq \hat{G}$, $X^\triangleleft = \{g \in G : (\forall \chi \in X) \left|\chi(g)\right| \leq \frac{1}{4}\}$.

**Definition [Vilenkin]**

A set $A \subseteq G$ is quasi-convex if $A^{\triangleright \triangleleft} = A$. $G$ is locally quasi-convex if it has a neighborhood base at its identity consisting of quasi-convex sets.
Observe that for each compact subset $K$ of $G$, the set $K_n = K \cup 2K \cup \cdots \cup nK$ is also compact, and $[K_n, 1/4] \subset [K, 1/4n]$. Thus, the sets $[K, 1/4], \ K \in \mathcal{K}(G)$, also form a neighborhood base of $\hat{G}$ at the trivial character.

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**Lemma [Banaszczyk]**

For each neighborhood $U$ of 0 in $G$, $U^\triangleright \in \mathcal{K}(\hat{G})$. 
In a different direction, we need the following definitions.

**Definition**

Let \((\mathbb{P}, \leq)\) be a partial ordered set. A subset \(D\) is **cofinal** in \(\mathbb{P}\) if \(D \subseteq \mathbb{P}\), and for each \(p \in \mathbb{P}\) there is \(d \in D\) such that \(p \leq d\). The **cofinality** of \(\mathbb{P}\), denoted \(\text{cof}(\mathbb{P})\), is the minimal cardinality of a cofinal set in \(\mathbb{P}\).
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**Definition**

Let $(\mathbb{P}, \leq)$ be a partial ordered set. A subset $D$ is **cofinal** in $\mathbb{P}$ if $D \subset \mathbb{P}$, and for each $p \in \mathbb{P}$ there is $d \in D$ such that $p \leq d$. The **cofinality** of $\mathbb{P}$, denoted $\text{cof}(\mathbb{P})$, is the minimal cardinality of a cofinal set in $\mathbb{P}$.

Given two partial ordered sets $(X, \leq)$ and $(Y, \preceq)$ we say that $(X, \leq)$ has **cofinality greater or equal than** $(Y, \preceq)$ when there is a map $\Phi : X \to Y$ which preserves the order and such that $\Phi(X)$ is cofinal in $Y$. **Cofinality equivalence** is defined accordingly.
For each $A \subseteq G$, $A^\triangleright$ is always a quasi-convex subset of $\hat{G}$. Thus, $\hat{G}$ is locally quasi-convex for all topological groups $G$. Moreover, local quasi-convexity is hereditary for arbitrary subgroups.
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**Lemma**

Let $G$ be a complete locally quasi-convex group. Let $\mathcal{N}_{\hat{G}}$ be the family of all neighborhoods of 0 in $\hat{G}$. Then:

- $(\mathcal{N}_{\hat{G}}, \supseteq)$ is cofinally equivalent to $(\mathcal{K}(G), \subseteq)$.
- $\chi(\hat{G}) = \text{cof}(\mathcal{K}(G))$. 

Salvador Hernández
Bounded sets in topological groups
[Proof]

Clearly, the polar map $\triangleright : \mathcal{K}(G) \to \mathcal{N}_{\hat{G}}$ is monotone and cofinal. Consider the other direction. Let $K \in \mathcal{K}(G)$, and take $U = K^\triangleright \in \mathcal{N}_{\hat{G}}$. By a Banaszczyk Lemma $U^\triangleright \in \mathcal{K}(\hat{G})$. Now,

$$K \subset K^{\triangleright \triangleleft} = U^\triangleleft = \mathcal{E}^{-1}[U^\triangleright \cap \mathcal{E}[G]].$$

Clearly $U^\triangleright \cap \mathcal{E}[G]$ is precompact and, since $\mathcal{E}$ is open and $G$ is complete, it follows that $\mathcal{E}^{-1}[U^\triangleright \cap \mathcal{E}[G]]$ is compact. Thus, the monotone map $\triangleleft : \mathcal{N}_{\hat{G}} \to \mathcal{K}(G)$ is also cofinal.
Clearly, the polar map \( \rhd : \mathcal{K}(G) \rightarrow \mathcal{N}_G \) is monotone and cofinal. Consider the other direction. Let \( K \in \mathcal{K}(G) \), and take \( U = K^{\rhd} \in \mathcal{N}_G \). By a Banaszczyk Lemma \( U^{\rhd} \in \mathcal{K}(\hat{G}) \). Now,

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Clearly \( U^{\rhd} \cap \mathcal{E}[G] \) is precompact and, since \( \mathcal{E} \) is open and \( G \) is complete, it follows that \( \mathcal{E}^{-1}[U^{\rhd} \cap \mathcal{E}[G]] \) is compact. Thus, the monotone map \( \triangleleft : \mathcal{N}_G \rightarrow \mathcal{K}(G) \) is also cofinal.

**Proposition**

Let \( G \) be subreflexive, and \( \mathcal{N}_G \) be the family of all neighborhoods of 0 in \( G \). Then:

1. \( (\mathcal{N}_G, \supset) \) is cofinally equivalent to \( (\mathcal{K}(\hat{G}), \subset) \).
2. \( \chi(G) = \text{cof}(\mathcal{K}(\hat{G})) \).
1. Introduction

2. Duality of abelian groups

3. Bounded sets in topological groups

4. Main results
Thus duality has taken us to the question of estimating $\text{cof}(\mathcal{K}(G))$ for topological abelian groups $G$. 
Assume that $G$ is a metrizable topological group, $\mathcal{V} = \{V_m\}_{m<\omega}$ is a neighborhood base of the identity $e$, $I$ is a dense subset of $G$ having minimal cardinality, and $\mathcal{B}$ denotes the collection of all precompact (or bounded) subsets of $G$. 
Assume that $G$ is a metrizable topological group, $\mathcal{V} = \{V_m\}_{m<\omega}$ is a neighborhood base of the identity $e$, $I$ is a dense subset of $G$ having minimal cardinality, and $\mathcal{B}$ denotes the collection of all precompact (or bounded) subsets of $G$.

If $[I]^*$ designates the collection of (non empty) finite subsets of $I$ and $\omega[I]^*$ the set of all functions $f : \omega \to [I]^*$,
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If $[I]^*$ designates the collection of (non empty) finite subsets of $I$ and $\omega[I]^*$ the set of all functions $f : \omega \rightarrow [I]^*$, then we have defined a map

$$\Psi_{\mathcal{V}} : \omega[I]^* \rightarrow \mathcal{B}$$

as follows

$$\Psi_{\mathcal{V}}(\alpha) = \bigcap_{m \in \omega} \left( \bigcup_{i \in \alpha(m)} (iV_m) \right).$$
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$$\Psi_{\mathcal{V}}(\alpha) = \bigcap_{m \in \omega} \left( \bigcup_{i \in \alpha(m)} i V_m \right).$$

If we consider the inclusion order in $\mathcal{B}$ and the pointwise inclusion order "$\subseteq$" in $\omega[I]^*$; that is, for $\alpha, \beta \in \omega[I]^*$, $\alpha \subseteq \beta$ means: $\alpha(n) \subseteq \beta(n)$ for all $n < \omega$. 
Lemma

The map \( \Psi \) is order preserving and \( \Psi(\omega [I]^*) \) is a cofinal subset of \( \mathcal{B} \).
Lemma
The map $\Psi_V$ is order preserving and $\Psi_V(\omega[I]^*)$ is a cofinal subset of $\mathcal{B}$.

Proof
It is clear that $\Psi_V$ preserves the order. Now, let $B \in \mathcal{B}$. Then, for all $m \in \omega$, there is $F_m \in [I]^*$ such that $B \subseteq \bigcup_{i \in F_m} iV_m$. Take $\alpha : \omega \longrightarrow [I]^*$ defined by $\alpha(m) = F_m$. Then $\Psi_V(\alpha) \supseteq B$. 
Lemma

The map $\Psi_{\mathcal{I}}$ is order preserving and $\Psi_{\mathcal{I}}(\omega[I]^*)$ is a cofinal subset of $\mathcal{B}$.

Proof

It is clear that $\Psi_{\mathcal{I}}$ preserves the order. Now, let $B \in \mathcal{B}$. Then, for all $m \in \omega$, there is $F_m \in [I]^*$ such that $B \subseteq \bigcup_{i \in F_m} iV_m$. Take $\alpha : \omega \rightarrow [I]^*$ defined by $\alpha(m) = F_m$. Then $\Psi_{\mathcal{I}}(\alpha) \supseteq B$.

Notice that for a bounded neighbourhood $U$ of the identity in $G$, the set

$$\mathcal{B}_U = \left\{ \bigcup_{i \in F} iU : F \in [I]^* \right\}$$

is cofinal in $\mathcal{B}$. As a consequence, if $G$ is locally bounded and $\kappa = |I|$, then $\mathcal{B}$ is cofinally equivalent to $[\kappa]^*$ and $\text{cof}(\mathcal{B}) = \kappa$. 

Salvador Hernández
Bounded sets in topological groups
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$$\Theta(\alpha)(n) = \max\{j : j \in \alpha(n)\}$$

establishes the cofinal equivalence between the sets $\omega[\omega]^*$ and $\omega\omega$, the later equipped with the canonical pointwise order. Hence, we shall take the set $\omega\omega$ (not $\omega[\omega]^*$) for the sake of simplicity.
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When $|I| = \omega$, in addition to the already defined map $\Psi_{\mathfrak{I}}$, it is also useful to consider the map

$$\Phi_{\mathfrak{I}} : \mathcal{B} \to \omega\omega$$

defined by the rule

$$\Phi_{\mathfrak{I}}(K)(m) := \min \left\{ n : K \subseteq \bigcup_{i \leq n} iV_m \right\}.$$
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defined by the rule

$$\Phi_V(K)(m) := \min \left\{ n : K \subseteq \bigcup_{i \leq n} iV_m \right\}.$$ 

$\Phi_V$ is order preserving and relates the confinality of $\mathcal{B}$ and $\omega\omega$. 
Lemma

If $G$ is metrizable and not locally bounded, then there is a neighborhood base $\mathcal{V} = \{U_m\}_{m < \omega}$ such that $\Phi_{\mathcal{V}}(\mathcal{B})$ is cofinal in $\omega\omega$. 

Proof

It suffices to show that there is a neighborhood base $\{U_n: n \in \omega\}$, at $e$, such that for each $f \in \omega\omega$, there is a compact $K \subseteq G$, with $f \leq \Phi_{\mathcal{V}}[K]$. 

Let $U_n, n \in \mathbb{N}$, be a descending neighborhood base at $e$ consisting of unbounded subsets. As each $U_n$ is not bounded, we may assume (by shrinking $U_{n+1}$ if needed) that there is no $m$ such that $U_n \subseteq \{1, \ldots, m\} \cdot U_{n+1}$. 

Given $f \in \omega\omega$, choose for each $n$ an element $x_n \in U_n \setminus \{1, \ldots, f(n)\} \cdot U_{n+1}$. As the original sequence $U_n$ was descending to $e$, \{x_n\} converges to $e$, and thus $K = \{x_n: n \in \mathbb{N}\} \cup \{e\}$ is the required compact subset of $G$. 

Salvador Hernández

Bounded sets in topological groups
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Salvador Hernández

Bounded sets in topological groups
Theorem

Let $G$ be a separable metrizable group. Then $B$ is cofinally equivalent to one of following ordered sets, $\{0\}$, $\omega$, or $\omega_\omega$, depending on whether $G$ is either trivial or bounded, locally bounded, or non locally bounded respectively.
Theorem

Let $G$ be a separable metrizable group. Then $\mathcal{B}$ is cofinally equivalent to one of following ordered sets, $\{0\}$, $\omega$, or $\omega \omega$, depending on whether $G$ is either trivial or bounded, locally bounded, or non locally bounded respectively. Hence, if the group $G$ is not locally bounded, it follows that $\text{cof}(\mathcal{B}) = \omega$. 
The extension of these results to arbitrary metrizable groups $G$ deals not only with the density $d(G)$ of the group but with another invariant cardinal of the group, named *local density* of $G$, which is defined by

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\text{ld}(G) = \min \{d(U) : U \text{ is a nbd of } e \text{ in } G\}.
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$$ld(G) = \min\{d(U) : U \text{ is a nbd of } e \text{ in } G\}.$$ 

We say that $G$ has stable density if

$$ld(G) = d(G).$$
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We say that $G$ has *stable density* if

$$\text{ld}(G) = d(G).$$

**Theorem**

Let $G$ be a metrizable group that is not locally bounded. Then

$$\text{cof}(\mathcal{B}) = \vartheta \cdot d(G) \cdot \text{cof}(\omega[\text{ld}(G)]).$$
The extension of these results to arbitrary metrizable groups $G$ deals not only with the density $d(G)$ of the group but with another invariant cardinal of the group, named *local density* of $G$, which is defined by

$$ld(G) = \min\{d(U) : U \text{ is a nbd of } e \text{ in } G\}.$$ 

We say that $G$ has *stable density* if

$$ld(G) = d(G).$$

**Theorem**

Let $G$ be a metrizable group that is not locally bounded. Then

$$\text{cof}(\mathcal{B}) = \omega \cdot d(G) \cdot \text{cof}(\omega[ld(G)]).$$

Using duality we obtain
Theorem

Let $G$ be a subreflexive group, such that the group $\Gamma = \hat{G}$ is metrizable. Then $\chi(G) = \text{cof}(B(\Gamma))$. Thus,

- If $\Gamma$ is precompact, then $\chi(G) = 1$, that is, $G$ is discrete.
- If $\Gamma$ is nonprecompact locally precompact, then $\chi(G) = d(\Gamma)$.
- If $\Gamma$ is not locally precompact, then $\chi(G) = d \cdot d(\Gamma) \cdot \text{cof}(\omega \text{ld}(\Gamma))$.
Theorem

Let $G$ be a subreflexive group, such that the group $\Gamma = \hat{G}$ is metrizable. Then $\chi(G) = \text{cof}(\mathcal{B}(\Gamma))$. Thus,

- If $\Gamma$ is precompact, then $\chi(G) = 1$, that is, $G$ is discrete.
- If $\Gamma$ is nonprecompact locally precompact, then $\chi(G) = d(\Gamma)$.
- If $\Gamma$ is not locally precompact, then $\chi(G) = \varnothing \cdot d(\Gamma) \cdot \text{cof}([\omega \text{l}(\Gamma)])$.

In particular, we deduce for free abelian topological groups

Corollary

Assume that $X$ is a $k_\omega$ space. Then

$$\chi(A(X)) = \varnothing \cdot \text{cof}(\omega[\kappa]),$$

where $\kappa = \sup\{w(K) : K \in \mathcal{K}(X)\}$. 
Theorem

Let $G$ be a locally quasi-convex $k_\omega$ group with $\{K_n : n \in \omega\}$ cofinal in $\mathcal{K}(G)$. Let

$$
\kappa = \sup_{n \in \mathbb{N}} w(K_n); \text{ and } \\
\lambda = \min\{\sup\{w(K) : K \in \mathcal{K}(G/H)\} : H \leq G \text{ compact}\}.
$$

- If $G$ is nondiscrete and locally compact, then $\chi(G) = \kappa$.
- If $G$ is not locally compact, then $\chi(G)$ is the maximum of $\varnothing$, $\kappa$, and $\operatorname{cof}(\omega[\lambda])$. 