The Cesàro operator on weighted $c_0$ spaces

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Joint work with A.A. Albanese and W.J. Ricker

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Aim of the lecture

**AIM**

Characterize the continuity, the compactness, the mean ergodicity and determine the spectrum of the Cesàro operator $C$ acting on the weighted Banach sequence space $c_0(w)$.

We report on joint work with Angela A. Albanese (Univ. Lecce, Italy) and Werner J. Ricker (Univ. Eichstaett, Germany).
Ernesto Cesàro (1859-1906)

The Cesàro operator on weighted $c_0$ spaces
The Cesàro operator on weighted $c_0$ spaces
The Cesàro operator $C$ is defined for a sequence $x = (x_n)_n \in \mathbb{C}^\mathbb{N}$ of complex numbers by

$$C(x) = \left( \frac{1}{n} \sum_{k=1}^{n} x_k \right)_n, \quad x = (x_n)_n \in \mathbb{C}^\mathbb{N}. $$

**Proposition.**

The operator $C : \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$ is a bicontinuous isomorphism of $\mathbb{C}^\mathbb{N}$ onto itself with

$$C^{-1}(y) = (ny_n - (n - 1)y_{n-1})_n, \quad y = (y_n)_n \in \mathbb{C}^\mathbb{N},$$

where we set $y_{-1} := 0$.

Recall that $\mathbb{C}^\mathbb{N}$ is a Fréchet space for the topology of coordinatewise convergence.
The discrete Cesàro operator on Banach sequence spaces

Theorem. Hardy. 1920.

Let $1 < p < \infty$. The Cesàro operator maps the Banach space $\ell^p$ continuously into itself, which we denote by $C^{(p)} : \ell^p \to \ell^p$, and
\[
\|C^{(p)}\| = p', \text{ where } \frac{1}{p} + \frac{1}{p'} = 1, \text{ for all } n \in \mathbb{N}.
\]

In particular, **Hardy’s inequality** holds:
\[
\|C^{(p)}\|_p \leq p'\|x\|_p, \quad x \in \ell^p.
\]

Clearly $C$ is not continuous on $\ell_1$, since $C(e_1) = (1, 1/2, 1/3, \ldots)$. 

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Proposition.
The Cesàro operators $C^{(\infty)} : \ell^\infty \to \ell^\infty$, $C^{(c)} : c \to c$ and $C^{(0)} : c_0 \to c_0$ are continuous, and $\|C^{(\infty)}\| = \|C^{(c)}\| = \|C^{(0)}\| = 1$.
Moreover, $\lim Cx = \lim x$ for each $x \in c$. 
X is a Hausdorff locally convex space (lcs).

\( \mathcal{L}(X) \) (resp. \( \mathcal{K}(X) \)) is the space of all continuous (resp. compact) linear operators on \( X \).

The resolvent set \( \rho(T, X) \) of \( T \in \mathcal{L}(X) \) consists of all \( \lambda \in \mathbb{C} \) such that \( R(\lambda, T) := (\lambda I - T)^{-1} \) exists in \( \mathcal{L}(X) \).

The spectrum of \( T \) is the set \( \sigma(T, X) := \mathbb{C} \setminus \rho(T, X) \). The point spectrum is the set \( \sigma_{pt}(T, X) \) of those \( \lambda \in \mathbb{C} \) such that \( T - \lambda I \) is not injective. The elements of \( \sigma_{pt}(T, X) \) are called eigenvalues of \( T \).
Notation:

\[ \Sigma := \{ \frac{1}{m} : m \in \mathbb{N} \} \text{ and } \Sigma_0 := \Sigma \cup \{0\}. \]

Proposition.

(i) \( \sigma(C; \mathbb{C}^\mathbb{N}) = \sigma_{pt}(C; \mathbb{C}^\mathbb{N}) = \Sigma. \)

(ii) Fix \( m \in \mathbb{N} \). Let \( x^{(m)} := (x_n^{(m)})_n \in \mathbb{C}^\mathbb{N} \) where \( x_n^{(m)} := 0 \) for \( n \in \{1, \ldots, m - 1\} \), \( x_m^{(m)} := 1 \) and \( x_n^{(m)} := \frac{(n-1)!}{(m-1)!(n-m)!} \) for \( n > m \). Then the eigenspace

\[ \text{Ker} \left( \frac{1}{m} I - C \right) = \text{span} \{ x^{(m)} \} \subseteq \mathbb{C}^\mathbb{N} \]

is 1-dimensional.

(i) \( \sigma(C; \ell^\infty) = \sigma(C; c_0) = \{ \lambda \in \mathbb{C} \mid |\lambda - \frac{1}{2}| \leq \frac{1}{2} \} \).

(ii) \( \sigma_{pt}(C; \ell^\infty) = \{(1, 1, 1, \ldots)\} \).

(iii) \( \sigma_{pt}(C; c_0) = \emptyset \).

Let $1 < p < \infty$ and $1/p + 1/p' = 1$.

(i) $\sigma(C; \ell^p) = \{\lambda \in \mathbb{C} \mid |\lambda - \frac{p'}{2}| \leq \frac{p'}{2}\}$.

(ii) $\sigma_{pt}(C; \ell^p) = \emptyset$.

In particular, $C$ is not compact in the spaces $\ell^p, 1 < p \leq \infty$, or in the space $c_0$. 
The space $c_0(w)$

- Let $w = (w(n))_{n=1}^{\infty}$ be a bounded, strictly positive sequence. Define

  
  
  \[
  c_0(w) := \left\{ x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} : \lim_{n \to \infty} w(n)|x_n| = 0 \right\},
  \]

  equipped with the norm $\|x\|_{0,w} := \sup_{n \in \mathbb{N}} w(n)|x_n|$ for $x \in c_0(w)$.

- $c_0(w)$ is isometrically isomorphic to $c_0$ via the linear multiplication operator $\Phi_w : c_0(w) \to c_0$ given by

  \[
  x = (x_n)_{n \in \mathbb{N}} \to \Phi_w(x) := (w(n)x_n)_{n \in \mathbb{N}}. \tag{2}
  \]

- We are interested in the case when $\inf_{n \in \mathbb{N}} w(n) = 0$. Otherwise $c_0(w) = c_0$ with equivalent norms.
Theorem.

Let \( w \) be a bounded, strictly positive sequence. The Cesàro operator \( C^{(0,w)} \in \mathcal{L}(c_0(w)) \) if and only if

\[
\left\{ \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \right\}_{n \in \mathbb{N}} \in \ell_{\infty}.
\]

Moreover, \( \|C^{(0,w)}\| \geq 1. \)

If \( w \) is decreasing, then (3) is satisfied and \( \|C^{(0,w)}\| = 1. \)
Theorem.

Let \( w \) be a bounded, strictly positive sequence. The following conditions are equivalent.

(a) \( C^{(0, w)} \) is weakly compact.

(b) \( C^{(0, w)} \) is compact.

(c) The sequence

\[
\left\{ \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \right\}_{n \in \mathbb{N}} \in c_0.
\]
Let $w = (w(n))_{n=1}^{\infty}$ be two strictly positive sequences. Let $T_w : \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$ denote the linear operator given by

$$T_w x := \left( \frac{w(n)}{n} \sum_{k=1}^{n} \frac{x_k}{w(k)} \right)_{n \in \mathbb{N}}, \quad x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}. \quad (5)$$

Then $\Phi_w C = T_w \Phi_v$. Therefore, the Cesàro operator $C$ maps $c_0(w)$ continuously (resp., compactly) into $c_0(w)$ if and only if the operator $T_w \in \mathcal{L}(c_0)$ (resp., $T_w \in \mathcal{K}(c_0)$).
Continuity of $C$ on $c_0(w)$. A classical lemma

**Lemma. Banach’s Book.**

Let $A = (a_{nm})_{n,m \in \mathbb{N}}$ be a matrix with entries from $\mathbb{C}$ and $T : \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$ be the linear operator defined by

$$Tx := \left( \sum_{m=1}^{\infty} a_{nm}x_m \right)_{n \in \mathbb{N}}, \quad x = (x_n)_{n \in \mathbb{N}},$$

interpreted to mean that $Tx$ exists in $\mathbb{C}^\mathbb{N}$ for every $x \in \mathbb{C}^\mathbb{N}$. Then $T \in \mathcal{L}(c_0)$ if and only if the following two conditions are satisfied:

(i) $\lim_{n \to \infty} a_{nm} = 0$ for each fixed $m \in \mathbb{N}$;

(ii) $\sup_{n \in \mathbb{N}} \sum_{m=1}^{\infty} |a_{nm}| < \infty$.

In this case, $\|T\| = \sup_{n \in \mathbb{N}} \sum_{m=1}^{\infty} |a_{nm}|$. 

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Examples

Let \( w(2n + 1) = \frac{1}{n+1} \) for \( n \geq 0 \) and \( w(2n) = 2^{-n} \) for \( n \geq 1 \). Clearly \( \lim_{n \to \infty} w(n) = 0 \), but \( C \) does not act continuously in \( c_0(w) \).

Let \( \alpha > 0 \) and \( w(n) := \frac{1}{n^\alpha} \) for all \( n \in \mathbb{N} \). Since \( w \) is decreasing, \( C^{(0,w)} \in \mathcal{L}(c_0(w)) \). But \( C^{(0,w)} \) is not compact, since

\[
\frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} = \frac{1}{n^{\alpha+1}} \sum_{k=1}^{n} k^\alpha \geq \frac{1}{n^{\alpha+1}} \sum_{k=1}^{n} \int_{k-1}^{k} x^\alpha \, dx
\]

\[
= \frac{1}{n^{\alpha+1}} \int_{0}^{n} x^\alpha \, dx = \frac{1}{\alpha + 1}.
\]
Examples of compact operators Cesàro operators on $C^{(0,w)}$

**Proposition.**

Let $w$ be bounded, strictly positive and satisfy

$$\limsup_{n \to \infty} \frac{w(n+1)}{w(n)} \in [0,1),$$

then $C^{(0,w)} \in K(c_0(w))$.

Moreover, $\sigma_{pt}(C^{(0,w)}) = \Sigma; \quad \sigma(C^{(0,w)}) = \Sigma_0$.

One checks that that the condition (4) is valid to prove compactness.
Examples of compact operators Cesàro operators on $C^{(0,w)}$

- $C^{(0,w)} \in \mathcal{K}(c_0(w))$ for the following sequences:

  1. $w(n) := a^{-\alpha n}, n \in \mathbb{N}$, with $a > 1$, $\alpha_n \uparrow \infty$ and $\lim_{n \to \infty} (\alpha_n - \alpha_{n-1}) = \infty$.

  2. $w(n) := \frac{n^\alpha}{a^n}$ for $n \in \mathbb{N}$, where $a > 1$ and $\alpha \in \mathbb{R}$.

  3. $w(n) := \frac{a^n}{n!}$ for $n \in \mathbb{N}$, where $a \geq 1$.

  4. $w(n) := n^{-n}$ for $n \in \mathbb{N}$.

- Let $w(n) := e^{-\sqrt{n}}$ or $w(n) := e^{-(\log n)^\beta}$, $\beta > 1$, for $n \in \mathbb{N}$. Then $\lim_{n \to \infty} \frac{w(n+1)}{w(n)} = 1$, but $C \in \mathcal{K}(c_0(w))$. 

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Given a bounded, strictly positive sequence \( w \), let

\[
S_w := \{ s \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{1}{n^s w(n)} < \infty \}.
\]

In case \( S_w \neq \emptyset \) we define \( s_0 := \inf S_w \).

Moreover, let

\[
R_w := \{ t \in \mathbb{R} : \lim_{n \to \infty} n^t w(n) = 0 \}.
\]

In case \( R_w \neq \mathbb{R} \) we define \( t_0 := \sup R_w \). If \( R_w = \mathbb{R} \) we set \( t_0 = \infty \).

Recall \( \Sigma := \{ \frac{1}{m} : m \in \mathbb{N} \} \) and \( \Sigma_0 := \Sigma \cup \{0\} \).
Theorem.

Let \( w \) be a bounded, strictly positive sequence such that \( C^{(0,w)} \in \mathcal{L}(c_0(w)) \).

(1) The following inclusion holds:

\[
\Sigma_0 \subseteq \sigma(C^{(0,w)}).
\]

(2) Let \( \lambda \notin \Sigma_0 \). Then \( \lambda \in \rho(C^{(0,w)}) \) if and only if both of the conditions

(i) \( \lim_{n \to \infty} \frac{w(n)}{n^{1-\alpha}} = 0 \), and

(ii) \( \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{m=1}^{n-1} \frac{w(n)n^\alpha}{w(m)m^\alpha} < \infty \),

are satisfied, where \( \alpha := \text{Re} \left( \frac{1}{\lambda} \right) \).
(3) Suppose that $R_w \neq \mathbb{R}$, i.e., $t_0 < \infty$. Then we have the inclusions

$$\left\{ \frac{1}{m} : m \in \mathbb{N}, 1 \leq m < t_0 + 1 \right\} \subseteq \sigma_{pt}(C^{(0, w)}) \subseteq \left\{ \frac{1}{m} : m \in \mathbb{N}, 1 \leq m \leq t_0 + 1 \right\}.$$

In particular, $\sigma_{pt}(C^{(0, w)})$ is a proper subset of $\Sigma$.

If $R_w = \mathbb{R}$, then

$$\sigma_{pt}(C^{(0, w)}) = \Sigma.$$
Some ingredients of the proof of the main result

First ingredient.

The dual operator.

The dual operator \((C^{(0,w)})' \in \mathcal{L}(\ell_1(w^{-1}))\) satisfies \(||(C^{(0,w)})'|| = ||C^{(0,w)}||\) and it is given by

\[
(C^{(0,w)})' y = \left( \sum_{k=n}^{\infty} \frac{y_k}{k} \right)_{n \in \mathbb{N}}, \quad y = (y_n)_{n \in \mathbb{N}} \in \ell_1(w^{-1}).
\]

It satisfies \(0 \not\in \sigma_{pt}((C^{(0,w)})')\) and \(\Sigma \subseteq \sigma_{pt}((C^{0,w})')\).

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First ingredient.
The dual operator.

Proposition.
If $S_w \neq \emptyset$, then the dual operator $(C^{(0,w)})'$ of $C^{(0,w)}$ satisfies

$$\left\{ \lambda \in \mathbb{C}: \left| \lambda - \frac{1}{2s_0} \right| < \frac{1}{2s_0} \right\} \cup \Sigma \subseteq \sigma_{pt}((C^{(0,w)})'), \quad \text{and}$$

$$\sigma_{pt}((C^{(0,w)})') \setminus \Sigma_0 \subseteq \left\{ \lambda \in \mathbb{C}: \left| \lambda - \frac{1}{2s_0} \right| \leq \frac{1}{2s_0} \right\}.$$
Some ingredients of the proof of the main result

Second ingredient.

A result of Reade (1985)

For \( n \in \mathbb{N} \) the \( n \)-th row of the matrix for \((C - \lambda I)^{-1}: \mathbb{C}^N \rightarrow \mathbb{C}^N \) (\( \lambda \not\in \Sigma_0 \)) has the entries

\[
\frac{-1}{n\lambda^2 \prod_{k=m}^{n} \left(1 - \frac{1}{\lambda k}\right)}, \quad 1 \leq m < n,
\]

\[
\frac{n}{1 - n\lambda} = \frac{1}{1/n - \lambda}, \quad m = n,
\]

and all the other entries in row \( n \) are equal to 0. Therefore

\[
(C - \lambda I)^{-1} = D_{\lambda} - \frac{1}{\lambda^2} E_{\lambda},
\]

where the \( D \) is a diagonal operator and \( E_{\lambda} = (e_{nm})_{n,m\in\mathbb{N}} \) is a lower triangular matrix.
Some ingredients of the proof of the main result

Third ingredient.
A technical lemma improving Reade (1985)

Lemma.
Let $\lambda \in \mathbb{C} \setminus \Sigma_0$ and set $\alpha := \text{Re} \left( \frac{1}{\lambda} \right)$. Then there exist constants $d > 0$ and $D > 0$ (depending on $\alpha$) such that

$$\frac{d}{n^\alpha} \leq \prod_{k=1}^{n} \left| 1 - \frac{1}{k\lambda} \right| \leq \frac{D}{n^\alpha}, \quad n \in \mathbb{N}.$$
Proposition.

Let $w$ be a strictly positive, decreasing sequence.

(i)\[ \sigma(C^{(0,w)}) \subseteq \left\{ \lambda \in \mathbb{C}: \left| \lambda - \frac{1}{2} \right| \leq \frac{1}{2} \right\}. \] (7)

(ii) If $S_w \neq \emptyset$, then

\[ \left\{ \lambda \in \mathbb{C}: \left| \lambda - \frac{1}{2s_0} \right| \leq \frac{1}{2s_0} \right\} \cup \Sigma \subseteq \sigma(C^{(0,w)}). \] (8)
A sequence \( u = (u_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} \) is called \textbf{rapidly decreasing} if 

\[(n^m u_n)_{n \in \mathbb{N}} \in c_0 \text{ for every } m \in \mathbb{N}.
\] The space of all rapidly decreasing, \( \mathbb{C} \)-valued sequences is denoted by \( s \).

**Proposition.**

Let \( w \) be a bounded, strictly positive sequence. If \( C^{(0,w)} \in \mathcal{K}(c_0(w)) \), then

\[
\sigma_{pt}(C^{(0,w)}) = \Sigma \text{ and } \sigma(C^{(0,w)}) = \Sigma_0.
\]

Moreover, \( w \in s \) and \( S_w = \emptyset \).

**There exist weights** \( w \in s \) such that \( C^{(0,w)} \not\in \mathcal{K}(c_0(w)) \): Define \( w \) via 

\[
w(1) := 1 \text{ and } w(n) := \frac{1}{j^j} \text{ if } n \in \{2^{j-1} + 1, \ldots, 2^j\} \text{ for } j \in \mathbb{N}.
\]
Spectrum of $C^{(0,w)}$. Relevant examples

(1) $w(n) = \frac{1}{(\log(n+1))^\gamma}$ for $n \in \mathbb{N}$ with $\gamma \geq 0$. Then $s_0 = 1$ and $t_0 = 0$. We have

$$\sigma(C^{(0,w)}) = \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{2} \right| \leq \frac{1}{2} \right\},$$

and

$$\sigma_{pt}(C^{(0,w)}) = \emptyset \text{ if } \gamma = 0; \quad \sigma_{pt}(C^{(0,w)}) = \{1\} \text{ if } \gamma > 0.$$
Spectrum of $C^{(0,w)}$. Relevant examples

(2) $w(n) = \frac{1}{n^\beta}$ for $n \in \mathbb{N}$ with $\beta > 0$. Then $t_0 = \beta$ and

$$\left\{ \lambda \in \mathbb{C}: \left| \lambda - \frac{1}{2(\beta + 1)} \right| \leq \frac{1}{2(\beta + 1)} \right\} \cup \Sigma = \sigma(C^{(0,w)}), \quad \text{and}$$

$$\left\{ \frac{1}{m} : m \in \mathbb{N}, \ 1 \leq m < \beta + 1 \right\} = \sigma_{pt}(C^{(0,w)}).$$
Spectrum of $C^{(0, w)}$. Relevant examples

Checking the examples requires the following technical result:

**Lemma.**

Let $\alpha$ be a real number with $\alpha < 1$. Then

$$\sup_{n \in \mathbb{N}} \frac{1}{n^{1-\alpha}} \sum_{m=1}^{n-1} \frac{1}{m^\alpha} < \infty.$$
Power bounded operators

An operator $T \in \mathcal{L}(X)$ is said to be power bounded if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$.

If $X$ is a Banach space, an operator $T$ is power bounded if and only if $\sup_n \|T^n\| < \infty$.

If $X$ is a Fréchet space, an operator $T$ is power bounded if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ under $T$ are bounded. This is a consequence of the uniform boundedness principle.
Mean ergodic properties. Definitions

For $T \in \mathcal{L}(X)$, we set $T[n] := \frac{1}{n} \sum_{m=1}^{n} T^m$.

Mean ergodic operators

An operator $T \in \mathcal{L}(X)$ is said to be mean ergodic if the limits

$$Px := \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} T^m x, \quad x \in X,$$

exist in $X$.

If $T$ is mean ergodic, then one then has the direct decomposition

$$X = \text{Ker}(I - T) \oplus (I - T)(X).$$
Uniformly mean ergodic operators

If \( \{ T[n] \}_{n=1}^\infty \) happens to be convergent in \( \mathcal{L}_b(X) \) to \( P \in \mathcal{L}(X) \), then \( T \) is called *uniformly mean ergodic*.


Let \( T \) a (continuous) operator on a Banach space \( X \) which satisfies
\[
\lim_{n \to \infty} \| T^n / n \| = 0.
\]
The following conditions are equivalent:

1. \( T \) is uniformly mean ergodic.
2. \((I - T)(X)\) is closed.
Hypercyclic operator

$T \in \mathcal{L}(X)$, with $X$ separable, is called hypercyclic if there exists $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}_0\}$ is dense in $X$.

Supercyclic operator

If, for some $z \in X$, the projective orbit $\{\lambda T^n z : \lambda \in \mathbb{C}, \ n \in \mathbb{N}_0\}$ is dense in $X$, then $T$ is called supercyclic.

Clearly, hypercyclicality always implies supercyclicality.
Proposition.

- The Cesàro operator $C: \mathbb{C}^\mathbb{N} \rightarrow \mathbb{C}^\mathbb{N}$ is power bounded, uniformly mean ergodic and not supercyclic.

- The Cesàro operator $C^{(p)}: \ell^p \rightarrow \ell^p, 1 < p < \infty$, is not power bounded, not mean ergodic and not supercyclic.

- The Cesàro operator $C^{(0)}: c_0 \rightarrow c_0$ is power bounded, not mean ergodic and not supercyclic.
Proposition.

Let \( w \) be a decreasing, strictly positive sequence. Then \( C^{(0,w)} \in \mathcal{L}(c_0(w)) \) is power bounded.

Moreover, the following assertions are equivalent:

(i) \( C^{(0,w)} \) is mean ergodic.

(iii) The weight \( w \) satisfies \( \lim_{n \to \infty} w(n) = 0 \).
Uniform mean ergodicity of $C$ on $c_0(w)$

Proposition.

Let $w$ be a decreasing, strictly positive sequence. Then $C^{(0,w)} \in \mathcal{L}(c_0(w))$ is uniformly mean ergodic if and only if $w$ satisfies both of the conditions

(i) $\lim_{n \to \infty} w(n) = 0$, and

(ii) $\sup_{n \in \mathbb{N}} w(n + 1) \sum_{m=1}^{n-1} \frac{1}{mw(m+1)} < \infty$. 

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Uniform mean ergodicity of $C$ on $c_0(w)$

**Proposition.**

If $w$ is a decreasing, strictly positive sequence such that $C^{(0,w)} \in \mathcal{K}(c_0(w))$, then $C^{(0,w)}$ is uniformly mean ergodic.

**Examples.**

(i) For $w(n) = \frac{1}{(\log(n+1))^\gamma}$ for $n \in \mathbb{N}$ with $\gamma \geq 1$, the operator $C^{(0,w)}$ is not compact, mean ergodic and not uniformly mean ergodic.

(ii) For $w(n) = \frac{1}{n^\beta}$ for $n \in \mathbb{N}$ with $\beta \geq 1$, the operator $C^{(0,w)}$ is uniformly mean ergodic but not compact.
Proposition.

Let \( w \) be a bounded, strictly positive sequence such that \( C^{(0,w)} \in \mathcal{L}(c_0(w)) \). Then \( C^{(0,w)} \) is not supercyclic and hence, also not hypercyclic.

This is a direct consequence of a general result by Ansari and Bourdon, since \( \sigma_{pt}((C^{(0,w)})') \) is infinite.


A. A. Albanese, J. Bonet, W. J. Ricker, Mean ergodicity and spectrum of the Cesàro operator on weighted $c_0$ spaces, Preprint (2015).