Atomic decompositions in locally convex spaces

Jornadas de Análisis Matemático

Alicante

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Atomic Decompositions have been investigated by **Carando, Casazza, Gröchenig, Lassalle, Korobeĭnik, Schmidberg, Taskinen and others**.

Outline

Definition

Let E be a Hausdorff locally convex space.

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Let $\{x_j\}_{j=1}^{\infty} \subset E$ and let $\{x_j'\}_{j=1}^{\infty} \subset E'$, we say that $\left(\{x_j'\}, \{x_j\}\right)$ is an *atomic decomposition of E* if

$$x = \sum_{j=1}^{\infty} x_j'(x)\, x_j, \qquad \text{for all } x \in E,$$

the series converging in E.

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We denote by ω the space $\mathbb{K}^{\mathbb{N}}$ endowed by the product topology.
A sequence space $\mathbb{K}^{(\mathbb{N})} \subset \bigwedge \subset \mathbb{K}^{\mathbb{N}}$ is a lcs such that there exists a continuous inclusion in ω.

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- Let $\{e_i\}_{i=1}^{2}$ be an orthonormal basis of a two-dimensional vector space V with inner product, then

$$x_1 = e_1,\ x_2 = e_1 - e_2,\ x_3 = e_1 + e_2$$
$$x_1' = \frac{1}{3}e_1, x_2' = \frac{1}{3}e_1 - \frac{1}{2}e_2, x_3' = \frac{1}{3}e_1 + \frac{1}{2}e_2$$

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  is an atomic decomposition of V.

- Let $\{e_i\}_{i=1}^\infty$ an orthonormal basis of Hilbert space $\mathcal{H}$, then

$$\begin{aligned} \{x_j\}_{j=1}^\infty &= \{e_1, e_1, e_2, e_3, \ldots\} \\ \{x_j'\}_{j=1}^\infty &= \left\{\frac{1}{2}e_1, \frac{1}{2}e_1, e_2, e_3, \ldots\right\} \end{aligned}$$

  is an atomic decomposition of $\mathcal{H}$.

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- Let E be a lcs with a Schauder basis $\{e_j\}_{j=1}^{\infty} \subset E$, the coefficient functionals will be denoted by $\{e_j'\}_{j=1}^{\infty} \subset E'$. If $\left(\{e_j'\}, \{e_j\}\right)$ is a Schauder basis in a lcs E, then it is an atomic decomposition for E.

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- Let E be a lcs with a Schauder basis $\{e_j\}_{j=1}^{\infty} \subset E$, the coefficient functionals will be denoted by $\{e'_j\}_{j=1}^{\infty} \subset E'$. If $\left(\{e'_j\}, \{e_j\}\right)$ is a Schauder basis in a lcs E, then it is an atomic decomposition for E.
- Let E be a lcs and let $P : E \to E$ be a continuous linear projection. If $\left(\{x'_j\}, \{x_j\}\right)$ is an atomic decomposition for E, then $\left(\{P'\left(x'_j\right)\}, \{P\left(x_j\right)\}\right)$ is an atomic decomposition for $P\left(E\right)$.

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Barrelled locally convex spaces are those satisfying the uniform boundedness principle (Banach-Steinhaus' Theorem).

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Let E be a barrelled and complete lcs, the following are equivalent:

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Let E be a barrelled and complete lcs, the following are equivalent:

1. *E admits an atomic decomposition.*
2. *E is isomorphic to a complemented subspace of a complete sequence space with the canonical unit vectors as Schauder basis.*

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1. *E admits an atomic decomposition.*
2. *E is isomorphic to a complemented subspace of a complete sequence space with the canonical unit vectors as Schauder basis.*
3. *E is isomorphic to a complemented subspace of a complete sequence space with Schauder basis.*

Space structure

Let $\{x_j\} \subset E$ be a sequence of non-zero elements.

$$\bigwedge := \left\{ \alpha = (\alpha_j)_j \in \omega : \sum_{j=1}^{\infty} \alpha_j x_j \text{ is convergent in } E \right\} \subset \omega,$$

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$$\mathcal{Q} = \left\{ q_p\left((\alpha_j)_j\right) := \sup_n p\left(\sum_{j=1}^{n} \alpha_j x_j\right), \text{ for all } p \in \mathcal{P} \right\}.$$

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Theorem

If E is a complete lcs then $(\bigwedge, \mathcal{Q})$ is a complete lcs.

Definition

A lcs E has the *bounded approximation property* (BAP) if there exists $(A_j)_{j\in J} \subset L(E,E)$ a net which is equicontinuous whit $\dim(A_j(E)) < \infty$ for every $j \in J$ and $\lim_{j\in J} A_j(x) = x$ for every $x \in E$. In other words, $A_j \to I$ in $L_s(E)$.

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Characterization (Bessaga, Pełczyński)

Let E be a separable Fréchet space, the following are equivalent:

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Characterization (Bessaga, Pełczyński)

Let E be a separable Fréchet space, the following are equivalent:

1. *The bounded approximation property holds in E.*
2. *E is isomorphic to a complemented subspace of a complete sequence space with Schauder basis.*

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Characterization (Bessaga, Pełczyński)

Let E be a separable Fréchet space, the following are equivalent:

1. *The bounded approximation property holds in E.*
2. *E is isomorphic to a complemented subspace of a complete sequence space with Schauder basis.*
3. *E admits an atomic decomposition.*

Outline

Shrinking Atomic Decompositions

We define a linear operator $T : E \to E$ as $T_n(x) := \sum_{j=n+1}^{\infty} x_j'(x)\, x_j$.

Definition

An atomic decomposition $\left(\{x_j'\}, \{x_j\}\right)$ is *shrinking* if for all $x' \in E'$, $\lim_{n\to\infty} x' \circ T_n = 0$ (the limit is taken in the topology $E_\beta' = (E', \beta(E', E))$.

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Characterization

The following are equivalent:

1. $\left(\{x_j'\}, \{x_j\}\right)$ is a shrinking atomic decomposition.

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Characterization

The following are equivalent:

1. $\left(\{x_j'\}, \{x_j\}\right)$ is a shrinking atomic decomposition.
2. $\left(\{x_j\}, \{x_j'\}\right)$ is an atomic decomposition for E_β'.

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We define a linear operator $T : E \to E$ as $T_n(x) := \sum_{j=n+1}^{\infty} x'_j(x)\, x_j$.

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An atomic decomposition $\left(\{x'_j\}, \{x_j\}\right)$ is *shrinking* if for all $x' \in E'$, $\lim_{n\to\infty} x' \circ T_n = 0$ (the limit is taken in the topology $E'_\beta = (E', \beta(E', E))$.

Characterization

The following are equivalent:

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3. For all $x' \in E'$, $\sum_{j=1}^{\infty} x'(x_j)\, x'_j$ is convergent in E'_β.

Boundedly Complete Atomic Decompositions

Definition

An atomic decomposition $\left(\{x_j'\}, \{x_j\}\right)$ is *boundedly complete* if for all $x'' \in E_\beta''$, the series $\sum_{j=1}^{\infty} x_j'(x'')\, x_j$ converges in E.

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Proposition

Let E be a barrelled lcs with a Schauder basis $\{x_j\}_j$; $\left(\{x_j'\}, \{x_j\}\right)$ is a boundedly complete atomic decomposition if and only if for every $(\alpha_j)_j \subset \mathbb{K}$ such that $\left(\sum_{j=1}^{k} \alpha_j x_j\right)_k$ is bounded, then $\sum_{j=1}^{\infty} \alpha_j x_j$ is convergent.

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Proposition

If $\left(\{x_j'\},\{x_j\}\right)$ is a boundedly complete atomic decomposition for a barrelled and complete lcs E with E_β'' barrelled, then E is complemented in its bidual E_β''.

Properties

Proposition

Let E be a lcs and let $\left(\{x_j'\}, \{x_j\}\right)$ be a shrinking atomic decomposition of E. Then $\left(\{x_j\}, \{x_j'\}\right)$ is a boundedly complete atomic decomposition of E'.

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Let E be a lcs and let $\left(\{x_j'\},\{x_j\}\right)$ be a shrinking atomic decomposition of E. Then $\left(\{x_j\},\{x_j'\}\right)$ is a boundedly complete atomic decomposition of E'.

Theorem

Let $\left(\{x_j'\},\{x_j\}\right)$ be an atomic decomposition of a (barrelled) lcs E which is shrinking and boundedly complete, then E is (semi-)reflexive.

Outline

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An atomic decomposition $\left(\{x_j'\}, \{x_j\}\right)$ for a lcs E is said to be *unconditional* if for every $x \in E$, its series expansion $\sum_{j=1}^{\infty} x_j'(x)\, x_j$ converges unconditionally, that is, $x = \sum_{j=1}^{\infty} x_j'(x)\, x_j$ with unconditional convergence.

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McArthur, Retherford, 1969

If a series $\sum_{j=1}^{\infty} x_j$ converges unconditionally, then, for every bounded sequence of scalars $\{a_j\}$, the series $\sum_{j=1}^{\infty} a_j x_j$ converges and the operator

$$\begin{array}{ccc} \ell_\infty & \longrightarrow & E \\ \{a_j\} & \longrightarrow & \sum_{j=1}^{\infty} a_j x_j; \end{array}$$

is a continuous linear operator.

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Outline

An atomic decomposition on the space $C^\infty(K)$

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The Fréchet space topology in $C^\infty(K)$ is defined by the seminorms:

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Remark

No system of exponencials can be a basis in $C^\infty([0,1])$.

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Theorem

Let $K \subset \mathbb{R}^p$ be a compact set which is the closure of its interior and let $(\{u_j\}, \{e^{ix\cdot\lambda^j}\})$ be an unconditional atomic decomposition of $C^{\infty}(K)$, then there exists a continuous linear extension operator $T : C^{\infty}(K) \to C^{\infty}(\mathbb{R}^p)$.

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Theorem

Let us assume that there exists a continuous linear extension operator $T : C^\infty(K) \to C^\infty(\mathbb{R}^p)$. Then there are sequences $(\lambda^j) \subset \mathbb{R}^p$ and $(u_j) \in C^\infty(K)'$ such that $\left(\{u_j\}, \left\{e^{2\pi i x\cdot\lambda^j}\right\}\right)$ is an atomic decomposition for $C^\infty(K)$.

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- Let $M > 0$ such that $K \subset [-M, M]^p$
- Choosing $\phi \in D([-2M, 2M]^p)$ such that $\phi \equiv 1$ on a neighborhood of $[-M, M]^p$.

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The atomic decomposition of $C^\infty(K)$ is shrinking and boundedly complete since $C^\infty(K)$ is a Montel space.

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Thanks for your attention