Some results on weak fuzzy normed spaces

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Background: fuzzy metric spaces

- In 1975, Kramosil and Michalek introduced the notion of fuzzy metric space by extending the concept of probabilistic metric space, given by Menger in 1942, to the fuzzy situation.
    - Menger: The probabilistic aproach to the notion of distance through distribution functions.
    - Kramosil y Michalek: The fuzzy aproach to the notion of distance by means of fuzzy sets.
- The fuzzy metric spaces of Kramosil and Michalek are equivalent to the Menger probabilistic spaces.

Continuous t-norms

Definition

A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:
(i) $*$ is associative and commutative;
(ii) $a * 1 = a$ for every $a \in [0,1]$;
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$;
(iv) $*$ is continuous.

Three paradigmatic examples of continuous t-norms are $\wedge$, $\cdot$ and $*_L$ (the Lukasiewicz t-norm), which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ and $a *_L b = \max\{a + b - 1, 0\}$, respectively.

Note that $*_L \leq \cdot \leq \wedge$. Actually, $* \leq \wedge$ for every continuous t-norm $*$.

Fuzzy metric spaces

Definition

A *fuzzy metric* on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, \infty)$ satisfying the following conditions for every $x, y, z \in X$:

(i) $M(x, y, 0) = 0$;

(ii) $M(x, y, t) = 1$ *for all* $t > 0 \Leftrightarrow x = y$;

(iii) $M(x, y, t) = M(y, x, t)$ for all $t > 0$;

(iv) $M(x, y, t + s) \geq M(x, z, t) * M(z, y, s),$ *for all* $t, s \geq 0$;

(v) $M(x, y, \_) : [0, \infty) \longrightarrow [0, 1]$ is left continuous.

The triple $(X, M, *)$ is a *fuzzy metric space* in the sense of Kramosil and Michalek.

In their original definition, Kramosil and Michalek also required the following condition:

$$\lim_{t \to \infty} M(x, y, t) = 1.$$

Fuzzy metric spaces

If $(M, *)$ is a fuzzy metric on X, then M induces a topology, τ_M, on X which has as a base the collection

$$\{B_M(x, r, t) : x \in X, 0 < r < 1, t > 0\}$$

where

$$B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Proposition

(a) *If (X, d) is a metric space then there exists a fuzzy metric on X, M_d, such that $\tau_d = \tau_{M_d}$.*

(b) *If $(M, *)$ is a fuzzy metric on X then τ_M is a metrizable topology on X.*

Fuzzy normed spaces

Several different notions of a fuzzy norm have been defined and studied in the literature.
Cheng and Morderson, 1994 ($* = \wedge$)

Definition

A *fuzzy norm* on a real vector space X is a pair $(N, *)$ such that $*$ is a continuous t-norm and N is a fuzzy set in $X \times [0, \infty)$ satisfying the following conditions for every $x, y \in X$:

(N1) $N(x, 0) = 0$;
(N2) $N(x, t) = 1 \text{ for all } t > 0 \Leftrightarrow x = 0$;
(N3) $N(cx, t) = N(x, t/ \mid c \mid)$ for all $c \neq 0$;
(N4) $N(x + y, t + s) \geq N(x, t) * N(y, s)$ for all $t, s \geq 0$;
(N5) $N(x, \_) : [0, \infty) \longrightarrow [0, 1]$ is left continuous;
(N6) $\lim_{t\to\infty} N(x, t) = 1$.

The triple $(X, N, *)$ is a *fuzzy normed space.*

Relationship between probabilistic normed spaces and fuzzy normed spaces

$D^+ = \{F : \mathbb{R} \to [0,1]$: F *is a distribution function* : $F(0) = 0\}$.
Let $H \in D^+$ given by $H(t) = 0$ if $t \leq 0$ and $H(t) = 1$ if $t > 0$.
Šerstnev, 1963.

Definition

A *Šerstnev probabilistic normed space* is a triple $(X, \nu, *)$ where X is a a real vector space, $*$ is a continuous t-norm and ν is a mapping from X into $\mathcal{D}^+$, called *Šerstnev probabilistic norm,* satisfying the following conditions for every $x, y \in X$:

(SN1) $\nu_x = H$ if and only if $x = 0$;
(SN2) $\nu_{cx}(t) = \nu_x(t/ \mid c \mid)$ for all $c \neq 0$;
(SN3) $\nu_{(x+y)}(t + s) \geq \nu_x(t) * \nu_y(s)$ for all $t, s \geq 0$.

A mapping $F : \mathbb{R} \to [0,1]$ is called a distribution function if it is non-decreasing, left continuous with $inf_{t \in \mathbb{R}} F(t) = 0$ and $sup_{t \in \mathbb{R}} F(t) = 1$.

Relationship between probabilistic normed spaces and fuzzy normed spaces

The class of fuzzy normed spaces is equivalent to the class of Šerstnev probabilistic normed spaces, in the sense that:

- for each fuzzy normed space $(X, N, *)$, the triple $(X, \nu_N, *)$ is a Šerstnev space where $(\nu_N)_x(t) = N(x, t)$ for all $t > 0$, and $(\nu_N)_x(t) = 0$ for all $t \leq 0$
- for each Šerstnev space if $(X, \nu, *)$, the triple $(X, N_\nu, *)$ is a fuzzy normed space, where
$$N_\nu(x, t) = \nu_x(t)$$
for all $x \in X$ and $t \geq 0$.

Examples of fuzzy normed spaces

Example

Let $(X, \|\cdot\|)$ be a normed space,

(a) Let $N_s : X \times [0, \infty) \to [0, 1]$ given by $N_s(x, 0) = 0$ for all $x \in X$ and

$$N_s(x, t) = \frac{t}{t + \|x\|},$$

for all $x \in X$ and $t > 0$. Then $(N_s, *)$ is a fuzzy norm on X, where $*$ is any continuous t-norm.

(b) Let $N_{01} : X \times [0, \infty) \to [0, 1]$ given by $N_{01}(x, t) = 0$ if $t \le \|x\|$ and $N_{01}(x, t) = 1$ if $t > \|x\|$. Then $(N_{01}, *)$ is a fuzzy norm on X, where $*$ is any continuous t-norm.

Topology induced by a fuzzy norm

Proposition

Let $(X, N, *)$ be a fuzzy normed space. Then $(X, M_N, *)$ is a fuzzy metric space in the sense of Kramosil and Michalev where

$$M_N(x, y, t) = N(x - y, t), \ \forall x, y \in X \ \forall t \geq 0.$$

We denote by τ_N the topology on X induced by the fuzzy metric M_N and $B_N(x, r, t) = B_{M_N}(x, r, t)$.

Proposition

If $(X, N, *)$ is a fuzzy normed space then (X, τ_N) is a metrizable topological vector space. If in addition $* = \wedge$, then (X, τ_N) is a metrizable locally convex space.

Relationship between normed spaces and fuzzy normed spaces

Proposition

Let $(X, \| \ \|)$ be a normed space. Then, $(X, N_s, *)$ and $(X, N_{01}, *)$ are fuzzy normed spaces where $*$ is any continuous t-norm and

$$\tau_{N_s} = \tau_{N_{01}} = \tau_{\| \ \|}$$

where $\tau_{\| \ \|}$ is the topology induced by the norm $\| \ \|$.

Question

Is normable the topology induced by a fuzzy norm?

Relationship between normed spaces and fuzzy normed spaces

Classical normed spaces are strictly included in the class of fuzzy normed spaces.

Example

Given $0 < p < 1$, let l_p be the linear space of all sequences $\mathbf{x} := (x_n)_n$ of real numbers such that $\sum_{n=1}^{\infty} |x_n|^p$ is a convergent series. The function $d_p : l_p \times l_p \to \mathbb{R}^+$ given by $d_p(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} |x_n - y_n|^p$ is a translation invariant metric such that (l_p, τ_{d_p}) is a topological vector space that it is not normable.
Let N be the fuzzy set in $l_p \times [0, \infty)$ given by $N(\mathbf{x}, 0) = 0$ for all $\mathbf{x} \in l_p$ and

$$N(\mathbf{x}, t) = \frac{t^p}{t^p + d_p(\mathbf{0}, \mathbf{x})},$$

for all $\mathbf{x} \in l_p$ and $t > 0$.
$(N, \cdot)$ is a fuzzy norm on l_p and $\tau_N = \tau_{d_p}$, i.e, (l_p, τ_{d_p}) is fuzzy normable.

Weak fuzzy norms

If condition N6 ($\lim_{t\to\infty} N(x,t) = 1$) is removed in the definition of fuzzy norm we say that $(N, *)$ is a *weak fuzzy norm* on X.

Definition

A *weak fuzzy norm* on a real vector space X is a pair $(N, *)$ such that $*$ is a continuous t-norm and N is a fuzzy set in $X \times [0, \infty)$ satisfying the following conditions for every $x, y \in X$:

(N1) $N(x, 0) = 0$;
(N2) $N(x, t) = 1 \text{ for all } t > 0 \Leftrightarrow x = 0$;
(N3) $N(cx, t) = N(x, t/ \mid c \mid)$ for all $c \neq 0$;
(N4) $N(x + y, t + s) \geq N(x, t) * N(y, s)$ for all $t, s \geq 0$;
(N5) $N(x, \_) : [0, \infty) \longrightarrow [0, 1]$ is left continuous;
The triple $(X, N, *)$ is a *weak fuzzy normed space*.

Motivation: The dual space of a fuzzy normed space

• C. Alegre, S. Romaguera, *The Hahn Banach Theorem for Fuzzy Normed Spaces Revisited*, AAA 2014.

Let $(X, N, \wedge)$ be a fuzzy normed space and let $(N_s, \wedge)$ be the standard fuzzy norm on $\mathbb{R}$.
Let

$$X^* = \{f : (X, \tau_N) \to (\mathbb{R}, \tau_{N_s}) \ : f \text{ is linear and continuous}\}$$

The pair $(N^*, \wedge)$ defined by $N^*(f, 0) = 0$ for all $f \in X^*$, and

$$N^*(f, t) = \sup\{\alpha \in [0, 1) : p^*_\alpha(f) < t\},$$

is a weak fuzzy norm on X^*.

For all $\alpha \in [0, 1)$, $x \in X$ and $f \in X^*$

$$p_\alpha(x) = \inf\{t > 0 : N(x, t) \geq \alpha\}$$

$$p^*_\alpha(f) = \sup\{|f(x)| : p_{1-\alpha}(x) \leq 1\}.$$

The dual space of a fuzzy normed space

Example

Let X be the linear space of all sequences $\mathbf{x} = (x_n)_n$ of real scalars. For each $\alpha \in (0,1)$ there exists $n \in \mathbb{N}$ such that $\alpha \in (\frac{n-1}{n}, \frac{n}{n+1}]$.
Let $p_0(\mathbf{x}) = 0$ and let

$$p_\alpha(\mathbf{x}) = max\{|x_1|, |x_2|, \dots, |x_n|\}$$

Let

$$N(\mathbf{x}, t) = sup\{\alpha \in [0,1) : p_\alpha(\mathbf{x}) < t\}$$

$(N, \wedge)$ is a fuzzy norm on X.
$(N^*, \wedge)$ is a weak fuzzy norm on X^* nevertheless it is not a fuzzy norm.
Indeed, if we consider the linear continuous function given by
$f(\mathbf{x}) = x_1 + x_2 + x_3$ then

$$N^*(f, t) \leq 1/2, \; for \; every \; t \geq 0$$

i.e., $lim_{n\to\infty} N^*(f, t) \neq 1$.

More examples of weak fuzzy normed spaces

• G. Beer, The structure of extended-real valued metric spaces, Set-Valued Var. Anal., 2013.

Let X be a set and let $d : X \times X \to \mathbb{R}^+ \cup \{\infty\}$. If d satisfies the conditions of a metric we say that d is an *extended metric.*

• G. Beer, Norms with infinite values, J. Convex. Anal., 2015.
• G. Beer, J. Vanderwerff, Estructural properties of extended normed spaces, Set-Valued Var. Anal., 2015.

Let X be a linear space and let $\| \ \| : X \to \mathbb{R}^+ \cup \{\infty\}$. If $\| \ \|$ satisfies the conditions of a norm we say that $\| \ \|$ is an *extended norm.*

More examples of weak fuzzy normed spaces

Example

Let $(X, \|\cdot\|)$ be an extended normed space,

(a) Let $N : X \times [0, \infty) \to [0, 1]$ given by $N(x, 0) = 0$ for all $x \in X$ and

$$N(x, t) = \frac{t}{t + \|x\|},$$

for all $x \in X$ and $t > 0$. Then $(N, *)$ is a weak fuzzy norm on X, where $*$ is any continuous t-norm. Note that if $\|x\| = \infty$, then $\lim_{t\to\infty} N(x, t) = 0$.

(b) Let $N : X \times [0, \infty) \to [0, 1]$ given by $N(x, t) = 0$ if $t \leq \|x\|$ and $N(x, t) = 1$ if $t > \|x\|$. Then $(N, *)$ is a weak fuzzy norm on X, where $*$ is any continuous t-norm. Note that if $\|x\| = \infty$, then $\lim_{t\to\infty} N(x, t) = 0$.

Topology induced by a weak fuzzy norm

Proposition

Let $(X, N, *)$ be a weak fuzzy normed space. Then $(X, M_N, *)$ is a fuzzy metric space in the sense of Kramosil and Michalek where

$$M_N(x, y, t) = N(x - y, t), \ \forall x, y \in X \ \forall t \geq 0.$$

This fuzzy metric induces a metrizable topology, τ_N, on X.

Proposition

Let $(X, N, *)$ be a weak fuzzy normed space and let $\mathcal{B}$ the family of open balls with center in the origin. Then

(a) *$\mathcal{B}$ is a filter basis*

(b) *$B_N(0, r, t)$ is balanced, for all $0 < r < 1$ and $t > 0$.*

(c) *If $* = \wedge$, then $B_N(0, r, t)$ is convex, for all $0 < r < 1$ and $t > 0$.*

(d) *$B_N(0, r, t)$ is an absorbent set if and only if $\lim_{t\to\infty} N(x, t) = 1$, for all $x \in X$.*

Characterizing metrizable topological vector spaces in terms of weak fuzzy norms

If (X, τ) is a topological vector space and $(N, *)$ is a weak fuzzy norm on X, we say that $(N, *)$ is compatible with τ if $\tau_N = \tau$.

Theorem

For a topological vector space (X, τ) the following conditions are equivalent:

(1) *(X, τ) is metrizable;*

(2) *there is a fuzzy norm $(N, \cdot)$ on X compatible with τ;*

(3) *there is a weak fuzzy norm $(N, \cdot)$ on X compatible with τ.*

Theorem

For a topological vector space (X, τ) the following conditions are equivalent:

(1) *(X, τ) is metrizable and locally convex;*

(2) *there is a fuzzy norm $(N, \wedge)$ on X compatible with τ;*

(3) *there is a weak fuzzy norm $(N, \wedge)$ on X compatible with τ.*

Characterizing locally bounded topological vector spaces in terms of weak fuzzy norms

Theorem

For a topological vector space (X, τ) the following conditions are equivalent:

(1) (X, τ) *is locally bounded;*

(2) *there is a fuzzy norm $(N, \cdot)$ on X compatible with τ that $\lim_{t\to\infty} N(x, t) = 1$ uniformly on an open ball centered at origin;*

(3) *there is a weak fuzzy norm $(N, \cdot)$ on X compatible with τ that $\lim_{t\to\infty} N(x, t) = 1$ uniformly on an open ball centered at origin.*

Characterizing normable topological vector spaces in terms of weak fuzzy norms

Corollary

For a topological vector space (X, τ) *the following conditions are equivalent:*

(1) (X, τ) *is normable;*

(2) *there is a fuzzy norm* $(N, \wedge)$ *on* X *compatible with* τ *that* $\lim_{t\to\infty} N(x, t) = 1$ *uniformly on an open ball centered at origin;*

(3) *there is a weak fuzzy norm* $(N, \wedge)$ *on* X *compatible with* τ *that* $\lim_{t\to\infty} N(x, t) = 1$ *uniformly on an open ball centered at origin.*