A comparative survey of methods based on Banach's contraction principle to analyze the cost of algorithms with two recurrence equations

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Section

Some authors have applied the Banach contraction principle equipped with an appropriate complexity space introduced by Schellekens, see [Sch95], [GarRomSch08] or [RomSchTiVal07], and bicomplete quasi-metrics [RoSaTi07], [RomTi09], [SaadVaCho09] to show the existence and uniqueness of solution for the recurrence equations of several well known algorithms.

To complement such works we have chosen a kind of algorithm composed by **two recurrences**, from [Atk96].

```
function P(n)
  if n > 0 then
    Q(n-1); C; P(n-1); C; Q(n-1)
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```

The execution times $S(n)$ and $T(n)$ of $P(n)$ and $Q(n)$, satisfy, at least approximately, the recurrences

$$S(n) = S(n-1) + 2T(n-1) + K_1,$$

and

$$T(n) = 2S(n-1) + 2T(n-1) + K_2,$$

for $n \in \mathbb{N}$, and with K_1, K_2, nonnegative constants.

Section

Based on the notion of a fuzzy quasi-metric space [KM75].

Definition ([ChoGraRa06], [GregRom04])

A fuzzy quasi-metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, \infty)$ such that for all $x, y, z \in X$:

- (KM1) $M(x, y, 0) = 0$.
- (KM2) $x = y$ if and only if $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$.
- (KM3) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s \geq 0$.
- (KM4) $M(x, y, \_) : [0, \infty) \to [0, 1]$ is left continuous.

Definition ([ChoGraRa06], [GregRom04])

A fuzzy quasi-metric space is a triple $(X, M, *)$ such that X is a set and $(M, *)$ is a fuzzy quasi-metric on X.

Based on the notion of a fuzzy quasi-metric space [KM75].
Based on the notions of G-Cauchy sequence and G-complete fuzzy metric [Grab88] to obtain a fuzzy version of Banach fixed point theorem.

Definition

A sequence $(x_n)_n$ in a fuzzy metric space $(X, M, *)$ is **Cauchy** provided that $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for each $t > 0$ and $p \in \mathbb{N}$.

Definition

A fuzzy metric space $(X, M, *)$ is **complete** provided that every Cauchy sequence in X is convergent.

([SehBha72])

A **B-contraction** *on a fuzzy metric space* $(X, M, *)$ *is a self-map* f *on* X *such that there is a constant* $k \in (0, 1)$*:*

$$M(f(x), f(y), kt) \geq M(x, y, t) \text{ for all } x, y \in X, t > 0.$$

Based on the notion of a fuzzy quasi-metric space [KM75].
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Thus, Grabiec's fixed point theorem can be formulated as follows.

Theorem ([Grab88])

Let $(X, M, *)$ be a G-complete fuzzy metric space such that $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. Then every B-contraction on X has a unique fixed point.

Quasi-metric generalizations of B-contraction and G-completeness were introduced in [RoSaTi07] to define G-bicompleteness.

Theorem ([RoSaTi07])

Let $(X, M, *)$ be a G-bicomplete fuzzy quasi-metric space such that $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. Then every B-contraction on X has a unique fixed point.

Although G-(bi)completeness is a very strong kind of completeness, we have the following nice fact for our approach.

Theorem ([RoSaTi07])

Each bicomplete non-Archimedean fuzzy quasi-metric space is G-bicomplete.

The following result implies that [RoSaTi07] Theorem applies to the standard fuzzy quasi-metric space of any bicomplete non-Archimedean quasi-metric space.

Theorem

Let (X, d) be a bicomplete non-Archimedean quasi-metric space. Then $(X, M_d, \wedge)$ is a G-bicomplete (non-Archimedean) fuzzy quasi-metric space such that $\lim_{t \to \infty} M_d(x, y, t) = 1$ for all $x, y \in X$.

The domain of words Σ^∞: all finite and infinite sequences (**words**) over an **alphabet** Σ, ordered by the so-called information order $\sqsubseteq$ on Σ^∞ , i.e., $x \sqsubseteq y \Leftrightarrow x$ **is a prefix of** y.

Example

$$S \to aRR \to b|a|T|RT \to \lambda$$

a, aa, aab, aaba, aabab, . . .
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Denote by $x \sqcap y$ **the longest common prefix** of x and y, and for each $x \in \Sigma^\infty$ denote by $\ell(x)$ the length of x.

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[Smyth88] introduced a non-Archimedean quasi-metric $d_\sqsubseteq$ on Σ^∞:

$$d_\sqsubseteq(x, y) = 0 \text{ if } x \sqsubseteq y, \text{ and } d_\sqsubseteq(x, y) = 2^{-\ell(x \sqcap y)} \text{ otherwise}$$

Its specialization order coincides with the order $\sqsubseteq$, and thus the quasi-metric space $(\Sigma^\infty, d_\sqsubseteq)$ preserves the information provided by $\sqsubseteq$.

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Its specialization order coincides with the order $\sqsubseteq$, and thus the quasi-metric space $(\Sigma^\infty, d_\sqsubseteq)$ preserves the information provided by $\sqsubseteq$.

Consequently $d_\sqsubseteq$ is a bicomplete non-Archimedean quasi-metric on Σ^∞.

The **product (fuzzy quasi-metric) space** of two fuzzy quasi-metric spaces is the fuzzy quasi-metric space $(X_1 \times X_2, M_1 \times M_2, *)$ such that

$$(M_1 \times M_2)((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t).$$

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If $(X_1, M_1, \wedge)$ and $(X_2, M_2, \wedge)$ are non-Archimedean, then $(X_1 \times X_2, M_1 \times M_2, \wedge)$ is non-Archimedean.

If $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are bicomplete, then $(X_1 \times X_2, M_1 \times M_2, *)$ is bicomplete.

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By applying the previous results and remarks to the standard fuzzy quasi-metric space of $(\Sigma^\infty, d_\sqsubseteq)$ when $* = \wedge$, we deduce:

Theorem

$(\Sigma^\infty \times \Sigma^\infty, M_{d_\sqsubseteq} \times M_{d_\sqsubseteq}, \wedge)$ *is a bicomplete non-Archimedean fuzzy quasi-metric space such that* $\lim_{t\to\infty} (M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})((x_1, x_2), (y_1, y_2), t) = 1$. *Therefore,* **every B-contraction on this space has a unique fixed point**.

We build a functional Φ suggested by S and T.

$$\Phi : \Sigma^\infty \times \Sigma^\infty \to \Sigma^\infty \times \Sigma^\infty,$$

given for each pair $x^1, x^2 \in \Sigma^\infty$, by $\Phi(x^1, x^2) = (u^1, u^2)$, where $(u^1)_0 = S(0), \quad (u^2)_0 = T(0)$, and

$$(u^1)_n = x^1_{n-1} + 2(x^2)_{n-1} + K_1,$$

$$(u^2)_n = 2(x^1)_{n-1} + 2(x^2)_{n-1} + K_2,$$

for all $n \in \mathbb{N}$ such that $n \leq (\ell(x^1) \wedge \ell(x^2)) + 1$.

We build a functional Φ suggested by S and T.
We show that Φ is a B-contraction of the G-bicomplete (non-Archimedean) fuzzy quasi-metric space $(\Sigma^\infty \times \Sigma^\infty, M_{d_\sqsubseteq} \times M_{d_\sqsubseteq}, \wedge)$.

Hence Φ **has a unique fixed point which is the solution of the recurrences** S **and** T.

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To model the algorithm iterations, one actually works on the set of all finite words, that endowed with the restriction of $(M_{d_\sqsubseteq}, \wedge)$ *provides a non-bicomplete non-Archimedean fuzzy quasi-metric space.*

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The product space $(\Sigma^F \times \Sigma^F, M_{d_\sqsubseteq} \times M_{d_\sqsubseteq}, \wedge)$ is also a non-bicomplete non-Archimedean fuzzy quasi-metric space. It is bicompletable and its bicompletion is a non-Archimedean fuzzy quasi-metric space.

In particular, for each pair x^1, x^2 in the set of finite words, the sequence of iterations converges, in $(\Sigma^\infty \times \Sigma^\infty, (M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})^i, \wedge)$, to the solution for the pair of recurrence equations S and T.

Section

Definition

The product quasi-metric space of two quasi-metric spaces (X, d) and (Y, e) is the quasi-metric space $(X \times Y, d \times e)$, where $d \times e$ is

$$(d \times e)((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) \vee e(y_1, y_2),$$

for all $(x_1, y_1), (x_2, y_2) \in X \times Y$.

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Definition

Given a quasi-metric d on X, then the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$, is also a quasi-metric on X, called the **conjugate** of d, and the function d^s defined by $d^s(x, y) = d(x, y) \vee d^{-1}(x, y)$ is a metric on X.

Definition

A quasi-metric space (X, d) is said to be **bicomplete** if (X, d^s) is a complete metric space.

By a **contraction map** on a quasi-metric space *(X, d)* we mean a self-map f of X such that $d(fx, fy) \leq kd(x, y)$ for all $x, y \in X$, where k is a constant with $0 < k < 1$. The number k is called a contraction constant for f.

The classical Banach contraction principle can be generalized to the quasi-metric setting as follows (see for instance Lemma 2.4 of [1]).

Theorem

Let f be a contraction map on a bicomplete quasi-metric space (X,d). Then, for each $x \in X$, the sequence of iterations $(f^n x)_{n \in \omega}$ is convergent in (X, d^s) to a point $x_0 \in X$ which is the unique fixed point of f.

[Sch95] **complexity (quasi-metric) space**, $(\mathcal{C}, d_{\mathcal{C}})$

$$\mathcal{C} = \left\{ f : \omega \to (0, \infty] : \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\},$$

and d_C is the quasi-metric on $\mathcal{C}$ given by

$$d_{\mathcal{C}}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \left(\left(\frac{1}{f(n)} - \frac{1}{g(n)} \right) \vee 0 \right)$$

for all $f, g \in \mathcal{C}$.

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for all $f, g \in \mathcal{C}$.
Consequence of Theorem 1, and Remark on p. 317 [RomSch99]:

Theorem

The complexity space $(\mathcal{C}, d_{\mathcal{C}})$ is bicomplete.

We construct a monotone increasing functional Φ, which is a contraction on $(\mathcal{C} \times \mathcal{C}, d_{\mathcal{C}} \times d_{\mathcal{C}})$.

Theorem

Let Φ be the functional on $\mathcal{C} \times \mathcal{C}$ defined by

$$\Phi(f,g)(0) = (S(0), T(0)),$$

and

$$\Phi(f,g)(n) = (pf(n-1) + qg(n-1) + K_1,$$
$$rf(n-1) + sg(n-1) + K_2),$$

for $n \in \mathbb{N}$ and $f, g \in \mathcal{C}$.

(cont...)

We construct a monotone increasing functional Φ, which is a contraction on $(\mathcal{C} \times \mathcal{C}, d_{\mathcal{C}} \times d_{\mathcal{C}})$.

Theorem

(...cont)

If $\alpha < 1$, *where*

$$\alpha = \frac{1}{2}\left(\frac{1}{p \wedge r} + \frac{1}{q \wedge s}\right),$$

then:

(1) Φ *is a monotone increasing contraction on* $(\mathcal{C} \times \mathcal{C}, d_{\mathcal{C}} \times d_{\mathcal{C}})$ *with contraction constant* α.

(2) Φ *has a* **unique fixed point** (f_0, g_0).

Section

Take as a starting point Ricarte and Romaguera Theorem 2.2 [RicRom13] new version of Matowski's theorem (based on the notion of a t-norm of Hadzić), which generalized Banach's contraction principle using a type of contractions that generalize Hicks' C-contraccions.

Theorem ([RicRom13])

. Let $(X, M, *)$ be a complete fuzzy metric space and $f : X \to X$ a self-map such that

$$M(x, t, y) > 1 - t \to M(fx, fy, \varphi(t)) > 1 - \varphi(t),$$

for all $x, y \in X$ and $t > 0$, where $\varphi : [0, \infty) \to [0, \infty)$ is a nondereasing function satisfying $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. Then f has a unique fixed point

A preorder on a nonempty set X is a reflexive and transitive binary relation $\preceq$ on X. A preordered fuzzy (quasi-)metric space is a 4-tuple $(X, M, \preceq, *)$ such that $(X, M, *)$ is a fuzzy (quasi)-metric space and $\preceq$ is a preorder on X.

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If $(M, *)$ is a fuzzy quasi-metric on X, the relation $\leq_M$ on X given by $x \leq_M y \Leftrightarrow M(x, y, t) = 1$ for all $t > 0$, is an order on X called the specialization order of $(M, *)$.

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Theorem ([CasRomTi14])

If the ordered fuzzy quasi-metric space $(X, M, \leq_M, *)$ is $\leq_M$-complete and $f : X \to X$ is a $\leq_M$-nondecreasing self-map such that there is $x_0 \in X$ satisfying $x_0 \leq_M fx_0$, then f has a fixed point.

To deduce the existence of the solution for the algorithm equations we shall use the following subset of the complexity space from [Sch95]:

$$\mathcal{C}_1 = \{f \in \mathcal{C} : f(n) \geq 1 \text{ for all } n \in \mathbb{N}\}$$

and the function $Q_{\mathcal{C}}$ defined in [RomTi09]:

$$Q_{\mathcal{C}}(f, g, t) = \sum_{k=n}^{\infty} 2^{-k} \left(\left(\frac{1}{g(k)} - \frac{1}{f(k)} \right) \vee 0 \right),$$

where $t \in (n-1, n]$, $n \in \mathbb{N}$.

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where $t \in (n-1, n]$, $n \in \mathbb{N}$.

Now we can construct a fuzzy set M_1 in $(\mathcal{C} \times \mathcal{C}) \times (\mathcal{C} \times \mathcal{C}) \times [0, \infty)$ as:

$M_1((f_1, g_1), (f_2, g_2), 0) = 0$ for all $f_i, g_i \in \mathcal{C}_1$,

$M_1((f_1, g_1), (f_2, g_2), t) = 1$ if $f_1 \leq f_2$ and $g_1 \leq g_2$ and $t > 0$, and

$M_1((f_1, g_1), (f_2, g_2), t) = 1 - [Q_{\mathcal{C}}(f_1, f_2, t) \vee Q_{\mathcal{C}}(g_1, g_2, t)]$ otherwise.

This set must be appropriate to define the following fuzzy quasi-metric space:

Lemma (Lemma 1 [CasRomTi14])

$(\mathcal{C}_1, M_1, \leq_{M_1}, *_L)$ *is a* $\leq_{M_1}$*-complete fuzzy quasi-metric space.*

So we will have to show that:

$$Q_{\mathcal{C}}(f_1, f_2, t+s) \leq Q_{\mathcal{C}}(f_1, f_3, t) + Q_{\mathcal{C}}(f_3, f_2, s)$$

$$Q_{\mathcal{C}}(g_1, g_2, t+s) \leq Q_{\mathcal{C}}(g_1, g_3, t) + Q_{\mathcal{C}}(g_3, g_2, s)$$

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and that the fuzzy quasi-metric space is $\leq_{M_1}$-complete.
We have the following consequence Theorem 6 of [CasRomTi14]

Theorem ([CasRomTi14])

If $\Phi : \mathcal{C}_1 \to \mathcal{C}_1$ is a $\leq_{M_1}$-nondecreasing map and there is $f_0 \in \mathcal{C}_1$ such that $f_0 \leq_M \Phi f_0$, then Φ has a fixed point.

One of the most interesting facts of this new approach is that the contraction condition of the preceding theorem is automatically satisfied whenever the self-map f is nondecreasing for the specialization order and $\varphi : [0, \infty) \to [0, \infty)$ verifies that $\varphi(t) > 0$ for all $t > 0$.

We will have to show that if $f_1 \leq f_2$ and $g_1 \leq g_2$ then $\Phi(f_1, g_1) \leq \Phi(f_2, g_2)$ and then by the preceding Theorem we will find that Φ has a fixed point, which is the solution to the system of recurrence equations.

Section

Review previous approaches and sketch the application of a new approach [CasRomTi14]:

- Product of fuzzy quasi-metric defined on the domain of words [CasRomTi10b].
- Product quasi-metric space of complexity space [CasRomTi10a].
- Preordered complete fuzzy quasi-metric space.