Some results on nonconvex minimization for quasi-metric spaces

ISWAT 2014, Valencia, September 2014

Carmen Alegre, Josefa Marín

Instituto Universitario de Matemática Pura y Aplicada
Departamento Matemática Aplicada
Universidad Politécnica de Valencia

Minimization problem

- (X, d) quasi-metric space

Minimization problem

- (X, d) quasi-metric space
- $f : (X, d) \to (-\infty, +\infty]$

Minimization problem

- (X, d) quasi-metric space
- $f : (X, d) \to (-\infty, +\infty]$
- What conditions must satisfy X and f so that f has minimum?

Minimization

Weierstrass theorem

Let (X, τ) be a topological space, and assume $f : (X, \tau) \to (-\infty, +\infty]$ is τ-lower semicontinuous. Suppose moreover there is $a > inf\ f$ such that $\{x \in X : f(x) \leq a\}$ is a compact set. Then f has minimum.

Weierstrass theorem

Let (X, τ) be a topological space, and assume $f : (X, \tau) \to (-\infty, +\infty]$ is τ-lower semicontinuous. Suppose moreover there is $a > \mathit{inf}\ f$ such that $\{x \in X : f(x) \leq a\}$ is a compact set. Then f has minimum.

- In many situations of interest in applications this theorem can not be applied

Minimization

Weierstrass theorem

Let (X, τ) be a topological space, and assume $f : (X, \tau) \to (-\infty, +\infty]$ is τ-lower semicontinuous. Suppose moreover there is $a > inf\ f$ such that $\{x \in X : f(x) \leq a\}$ is a compact set. Then f has minimum.

- In many situations of interest in applications this theorem can not be applied
  - Metric spaces are not in general locally compact

Minimization

Weierstrass theorem

Let (X, τ) be a topological space, and assume $f : (X, \tau) \to (-\infty, +\infty]$ is τ-lower semicontinuous. Suppose moreover there is $a > \mathit{inf}\ f$ such that $\{x \in X : f(x) \leq a\}$ is a compact set. Then f has minimum.

- In many situations of interest in applications this theorem can not be applied
  - Metric spaces are not in general locally compact
  - Variational principles: Compactness is replaced by completeness

- W. Takahashi, 1991

Theorem

Let (X, d) be a complete metric space and let $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function on X bounded from below. Suppose that for any $u \in X$ with $\inf_{x \in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and

$$f(v) + d(u, v) \leq f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x \in X} f(x) = f(x_0)$.

w-distance on a metric space

- O. Kada, T. Suzuki, W. Takahashi, 1996

w-distance on a metric space

- O. Kada, T. Suzuki, W. Takahashi, 1996

Definition

A w-**distance** on a metric space (X, d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(W1) $q(x, y) \leq q(x, z) + q(z, y)$ *for all* $x, y, z \in X$;
(W2) $q(x, \cdot) : X \to \mathbb{R}^+$ *is lower semicontinuous on* (X, τ_d) *for all* $x \in X$;
(W3) for each $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left.\begin{array}{l} q(x,y) \leq \delta \\ q(x,z) \leq \delta \end{array}\right\} \to d(y,z) \leq \varepsilon$$

w-distance on a metric space

- O. Kada, T. Suzuki, W. Takahashi, 1996

Definition

A w-**distance** on a metric space (X, d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(W1) $q(x, y) \leq q(x, z) + q(z, y)$ *for all* $x, y, z \in X$;
(W2) $q(x, \cdot) : X \to \mathbb{R}^+$ *is lower semicontinuous on* (X, τ_d) *for all* $x \in X$;
(W3) for each $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left.\begin{array}{l} q(x, y) \leq \delta \\ q(x, z) \leq \delta \end{array}\right\} \to d(y, z) \leq \varepsilon$$

- If d is a metric on X it is a w-distance on (X, d).

A minimization theorem for complete metric spaces by using w-distances

Theorem (O. Kada et alt.)

Let (X, d) be a complete metric space and let $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function on X bounded from below. Suppose that there exists a w-distance q on (X, d) such that for any $u \in X$ with $\inf_{x \in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and

$$f(v) + q(u, v) \leq f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x \in X} f(x) = f(x_0)$.

A minimization theorem for complete metric spaces by using w-distances

Theorem (O. Kada et alt.)

Let (X, d) be a complete metric space and let $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function on X bounded from below. Suppose that there exists a w-distance q on (X, d) such that for any $u \in X$ with $\inf_{x \in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and

$$f(v) + q(u, v) \leq f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x \in X} f(x) = f(x_0)$.

This theorem is used to generalize:

A minimization theorem for complete metric spaces by using w-distances

Theorem (O. Kada et alt.)

Let (X, d) be a complete metric space and let $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function on X bounded from below. Suppose that there exists a w-distance q on (X, d) such that for any $u \in X$ with $\inf_{x \in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and

$$f(v) + q(u, v) \leq f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x \in X} f(x) = f(x_0)$.

This theorem is used to generalize:

- Ekeland's variational principle

Definition (S. Park, 2001)

A w-**distance** on a quasi-metric space (X, d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(W1) $q(x, y) \leq q(x, z) + q(z, y)$ *for all* $x, y, z \in X$*;*
(W2) $q(x, \cdot) : X \to \mathbb{R}^+$ *is lower semicontinuous on* $(X, \tau_{d^{-1}})$ *for all* $x \in X$*;*
(W3) for each $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left.\begin{array}{l} q(x, y) \leq \delta \\ q(x, z) \leq \delta \end{array}\right\} \to d(y, z) \leq \varepsilon$$

Definition (S. Park, 2001)

A w-**distance** on a quasi-metric space (X, d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(W1) $q(x, y) \leq q(x, z) + q(z, y)$ *for all* $x, y, z \in X$;
(W2) $q(x, \cdot) : X \to \mathbb{R}^+$ *is lower semicontinuous on* $(X, \tau_{d^{-1}})$ *for all* $x \in X$;
(W3) for each $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left.\begin{array}{l} q(x,y) \leq \delta \\ q(x,z) \leq \delta \end{array}\right\} \to d(y,z) \leq \varepsilon$$

- If d is a quasi-metric on X it is not necessarily a w-distance on (X, d).

- C. Alegre, J. Marín, CITA 2014

Definition

A **modified** ***w*****-distance** or ***mw*****-distance** on a quasi-metric space (X, d) is a function $q : X \times X \rightarrow \mathbb{R}^+$ satisfying the following conditions:

- C. Alegre, J. Marín, CITA 2014

Definition

A **modified** ***w*****-distance** or *mw***-distance** on a quasi-metric space (X, d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(W1) $q(x, y) \leq q(x, z) + q(z, y)$ *for all* $x, y, z \in X$*;*
(W2) $q(x, \cdot) : X \to \mathbb{R}^+$ *is lower semicontinuous on* $(X, \tau_{d^{-1}})$ *for all* $x \in X$*;*

- C. Alegre, J. Marín, CITA 2014

Definition

A **modified** w**-distance** or mw**-distance** on a quasi-metric space (X, d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(W1) $q(x, y) \leq q(x, z) + q(z, y)$ *for all* $x, y, z \in X$*;*
(W2) $q(x, \cdot) : X \to \mathbb{R}^+$ *is lower semicontinuous on* $(X, \tau_{d^{-1}})$ *for all* $x \in X$*;*
(mW3) for each $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left.\begin{array}{l} q(y, x) \leq \delta \\ q(x, z) \leq \delta \end{array}\right\} \to d(y, z) \leq \varepsilon$$

Let (X, d) be a quasi-metric space:

Let (X, d) be a quasi-metric space:

(a) a **sequence** $(x_n)_{n\in\omega}$ in (X, d) is d-**Cauchy** (or **Cauchy**) if for each $\varepsilon > 0$ there exists $n_0 \in \omega$ such that $d(x_n, x_m) \leq \varepsilon$ whenever $n_0 \leq n \leq m$.

Convergence

Let (X, d) be a quasi-metric space:

(a) a **sequence** $(x_n)_{n\in\omega}$ in (X, d) is d-**Cauchy** (or **Cauchy**) if for each $\varepsilon > 0$ there exists $n_0 \in \omega$ such that $d(x_n, x_m) \leq \varepsilon$ whenever $n_0 \leq n \leq m$.

(b) a **quasi-metric space** (X, d) is d^{-1}**- complete** if every Cauchy sequence $(x_n)_{n\in\omega}$ in (X, d) converges with respect to the topology $\tau_{d^{-1}}$.

The results

Theorem

Let (X, d) be a d^{-1}-complete T_1 quasi-metric space, and let $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function on (X, d^{-1}), bounded from below. Suppose that there exists an *mw*-distance q on (X, d) such that for any $u \in X$ with $\inf_{x \in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$, $q(v, v) = 0$ and

$$f(v) + q(u, v) \leq f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x \in X} f(x) = f(x_0)$.

The results

Theorem

Let (X, d) be a d^{-1}-complete T_1 quasi-metric space, and let $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function on (X, d^{-1}), bounded from below. Suppose that there exists an *mw*-distance q on (X, d) such that for any $u \in X$ with $\inf_{x \in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$, $q(v, v) = 0$ and

$$f(v) + q(u, v) \leq f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x \in X} f(x) = f(x_0)$.

The results

Theorem

Let (X, d) be a d^{-1}-complete T_1 quasi-metric space. Let q be an *mw*-distance on (X, d) and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function on $(X, \tau_{d^{-1}})$, bounded from below.

The results

Theorem

Let (X, d) be a d^{-1}-complete T_1 quasi-metric space. Let q be an *mw*-distance on (X, d) and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function on $(X, \tau_{d^{-1}})$, bounded from below. Then, for any $\varepsilon > 0$ and $u \in X$ with $q(u, u) = 0$ and $f(u) \leq inf_{x \in X} f(x) + \varepsilon$, there exists $v \in X$ such that

Theorem

Let (X, d) be a d^{-1}-complete T_1 quasi-metric space. Let q be an *mw*-distance on (X, d) and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function on $(X, \tau_{d^{-1}})$, bounded from below. Then, for any $\varepsilon > 0$ and $u \in X$ with $q(u, u) = 0$ and $f(u) \leq inf_{x \in X} f(x) + \varepsilon$, there exists $v \in X$ such that

(i) $f(v) \leq f(u)$,

The results

Theorem

Let (X, d) be a d^{-1}-complete T_1 quasi-metric space. Let q be an *mw*-distance on (X, d) and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function on $(X, \tau_{d^{-1}})$, bounded from below. Then, for any $\varepsilon > 0$ and $u \in X$ with $q(u, u) = 0$ and $f(u) \leq inf_{x \in X} f(x) + \varepsilon$, there exists $v \in X$ such that

(i) $f(v) \leq f(u)$,

(ii) $q(u, v) \leq 1$,

The results

Theorem

Let (X, d) be a d^{-1}-complete T_1 quasi-metric space. Let q be an mw-distance on (X, d) and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function on $(X, \tau_{d^{-1}})$, bounded from below. Then, for any $\varepsilon > 0$ and $u \in X$ with $q(u, u) = 0$ and $f(u) \leq inf_{x \in X} f(x) + \varepsilon$, there exists $v \in X$ such that

(i) $f(v) \leq f(u)$,

(ii) $q(u, v) \leq 1$,

(iii) $f(v) < f(w) + \varepsilon q(v, w)$ for every $w \in X \setminus \{v\}$ with $q(w, w) = 0$.

Theorem

Let (X, d) be a d^{-1}-complete T_1 quasi-metric space. Let q be an mw-distance on (X, d) and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function on $(X, \tau_{d^{-1}})$, bounded from below. Then, for any $\varepsilon > 0$ and $u \in X$ with $q(u, u) = 0$ and $f(u) \leq inf_{x \in X} f(x) + \varepsilon$, there exists $v \in X$ such that

(i) $f(v) \leq f(u)$,

(ii) $q(u, v) \leq 1$,

(iii) $f(v) < f(w) + \varepsilon q(v, w)$ for every $w \in X \setminus \{v\}$ with $q(w, w) = 0$.

Observation

If $q = d$, (iii) implies that the function $g(x) = f(x) + \varepsilon q(v, x)$ has a minimum point v.

Corollary (strong form of the Ekeland variational priciple)

Let (X, d) be a complete metric space and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function on X, bounded from below. Then for any $\varepsilon > 0$ and $u \in X$ with $f(u) \leq inf_{x \in X} f(x) + \varepsilon$, there exists $v \in X$ such that

(i) $f(v) \leq f(u)$,

(ii) $d(u, v) \leq 1$,

(iii) $f(v) < f(w) + \varepsilon d(v, w)$ for every $w \in X \setminus \{v\}$.

In 2011 S. Cobzas proved a version of the Ekeland variational principle in the class of d^{-1}-complete T_1-quasi-metric spaces which generalizes the weak form of the classical variational principle. Its proof is based on the Brezis-Browder maximality principle. This result can be obtained as a corollary of the variational principle involving *mw*-distances.

Thanks for your attention